



Conjugacy problem in groups of oriented geometrizable 3-manifolds

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Abstract

The aim of this paper is to show that the fundamental group of an oriented 3-manifold which satisfies Thurston's geometrization conjecture has a solvable conjugacy problem. In other words, for any such 3-manifold M , there exists an algorithm which can decide for any couple of elements u, v of $\pi_1(M)$ whether u and v are in the same conjugacy class of $\pi_1(M)$ or not. More topologically, the algorithm decides for any two loops in M , whether they are freely homotopic or not.

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0. Introduction

Since the work of Dehn [4–6], the Dehn problems (more specifically the word problem and the conjugacy problem) have taken on major importance in the developments of combinatorial group theory all along the XXth century. But while elements of this theory provide straightforward solutions in special

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cases, there still remain important classes of groups for which actual techniques fail to give a definitive answer. This is particularly true when one considers the conjugacy problem. This last problem seems to be much more complicated than the word problem, and even very simple cases are still open or admit a negative answer.

Let us focus on fundamental groups of compact manifolds, a case which motivated the introduction of these problems by Dehn. Dehn has solved the word and conjugacy problems for groups of surfaces. It is well known that these two problems are in general insolvable in groups of n -manifolds for $n \geq 4$ (as a consequence of the facts that each finitely presented group is the fundamental group of some n -manifold for $n \geq 4$ fixed, and of the general unsolvability of these two problems for an arbitrary f.p. group). In case of dimension 3, not all f.p. groups occur as fundamental groups of 3-manifolds, but the problems are still open for this class, despite many improvements. To solve the word problem, one needs to suppose a further condition, namely Thurston's geometrization conjecture. In such a case, the automatic group theory gives a solution to the word problem, but fails to solve the conjugacy problem (cf. [8]). Some special cases nevertheless, admit, a solution. The small cancellation theory provides a solution for alternating links, and the biautomatic group theory provides solutions for hyperbolic manifolds and almost all Seifert fiber spaces; the best result we know applies to irreducible 3-manifolds with a non-empty boundary: they admit a locally $CAT(0)$ metric and hence their groups have a solvable conjugacy problem [2]. A major improvement has been the solution of Sela [34] for knot groups. In his paper, Sela conjectures that “the method seems to apply to all 3-manifolds satisfying Thurston's geometrization conjecture”. We have been inspired by his work, to show the main result of this paper:

Main Theorem. *The group of an oriented 3-manifold satisfying Thurston's geometrization conjecture has a solvable conjugacy problem.*

Let us first give precise definitions of the concept involved. Let $G = \langle X | R \rangle$ be a group given by a finite presentation. The *word problem* consists in finding an algorithm which decides for any couple of words u, v on the generators, if $u = v$ in G . The *conjugacy problem* consists in finding an algorithm which decides for any couple of words u, v on the generators, if u and v are conjugate in G (we shall write $u \sim v$), that is, if there exists $h \in G$, such that $u = h.v.h^{-1}$ in G . Such an algorithm is called a *solution* to the corresponding problem. It turns out that the existence of a solution to any of these problems does not depend upon the finite presentation of G involved. Novikov [26], has shown that a solution to the word problem does not exist in general. Since a solution to the conjugacy problem provides a solution to the word problem (to decide if $u = v$ just decide whether $uv^{-1} \sim 1$), the same conclusion applies to the conjugacy problem. Moreover, one can construct many examples of groups admitting a solution to the word problem, and no solution to the conjugacy problem (cf. [22]).

When one restricts one self to the fundamental group of a manifold, solving the word problem is equivalent to deciding for any couple of based point loops, whether they are homotopic or not with base point fixed, while solving the conjugacy problem is equivalent to deciding whether two loops are freely homotopic.

By a *3-manifold* we mean a compact connected *oriented* manifold with boundary, of dimension 3. According to the Moise theorem [23], we may indifferently use PL, smooth, or topological locally smooth, structures on such a 3-manifold.

A 3-manifold is said to satisfy Thurston's geometrization property (we will say that the manifold is *geometrizable*) if it decomposes in its canonical topological decomposition—along essential spheres

(Kneser–Milnor), disks (Dehn lemma) and tori (Jaco–Shalen–Johanson)—into pieces whose interiors admit a Riemannian metric complete and locally homogeneous. (In the following we shall speak improperly of a geometry on a 3-manifold rather than on its interior.)

Thurston has conjectured that all 3-manifolds are geometrizable. This hypothesis is necessary to our work, because otherwise one can classify neither 3-manifolds nor their group at present.

To solve the conjugacy problem, we will first use the classical topological decomposition, as well as the classification of geometrizable 3-manifolds, to reduce the conjugacy problem to the restricted case of closed irreducible 3-manifolds, which are either Haken (i.e. irreducible and containing a properly embedded 2-sided incompressible surface, cf. [16,17]), a Seifert fibered space [33,16], or modelled on *SOL* geometry (cf. [31]). That is, we show that if the group of any 3-manifold lying in such classes has a solvable conjugacy problem, then the same conclusion applies to any geometrizable 3-manifold (Lemma 1.4). The cases of Seifert fibered spaces and *SOL* geometry are rather easy, and we will only sketch solutions respectively in Sections 5.3 and 7; the interested reader can find detailed solutions in my PhD thesis [29]. The Haken case constitutes the essential difficulty, and will be treated in detail. We can further suppose that the manifold is not a torus bundle, because in such a case the manifold admits either a Seifert fibration or a geometric structure modelled on *SOL* geometry. As explained above, we solve the conjugacy problem in the group of a Haken closed manifold by essentially applying the strategy used by Sela to solve the case of knot groups.

Let us now focus on the case of a Haken closed manifold M . The JSJ decomposition theorem asserts that there exists a family of essential tori W , unique up to ambient isotopy, such that if one cuts M along W , one obtains pieces which are either Seifert fibered spaces or do not contain an essential torus (we say the manifold is atoroidal). According to Thurston's geometrization theorem [39], each atoroidal piece admits a hyperbolic structure with finite volume. This decomposition of M provides a decomposition of $\pi_1(M)$ as a fundamental group of a graph of groups, whose vertex groups are the π_1 of the pieces obtained, and edge groups are free abelian of rank 2. We then establish a conjugacy theorem (in the spirit of the conjugacy theorem for amalgams or HNN extensions), which characterizes conjugate elements in $\pi_1(M)$ (Theorem 3.1). This result, together with the algebraic interpretation of the lack of essential annuli in the pieces obtained (Proposition 4.1), and with a solution to the word problem, allows us to reduce the conjugacy problem in $\pi_1(M)$ into three algorithmic problems in the groups of the pieces obtained: the conjugacy problem, the boundary parallelism problem, and the 2-cosets problem (Theorem 4.1). Suppose M_1 is a piece, and T is a boundary subgroup of $\pi_1(M_1)$ (that is, $T = i_*(\mathbb{Z} \oplus \mathbb{Z}) \subset \pi_1(M_1)$ for some embedding $i : S^1 \times S^1 \rightarrow \partial M_1$). The *boundary parallelism problem* in $\pi_1(M_1)$ consists in finding for any element $\omega \in \pi_1(M_1)$ all the elements of T conjugate to ω in $\pi_1(M_1)$. Suppose T_1, T_2 are two boundary subgroups (possibly identical); the *2-cosets problem* consists in finding for any couples $\omega, \omega' \in \pi_1(M_1)$, all the elements $c_1 \in T_1, c_2 \in T_2$, such that $\omega = c_1 \cdot \omega' \cdot c_2$ in $\pi_1(M_1)$. We then solve these algorithmic problems, separately, in the Seifert case, and in the hyperbolic case, providing a solution to the conjugacy problem in $\pi_1(M)$.

In the Seifert case a solution can be easily established, by using the existence of a normal infinite cyclic subgroup $N \subset \pi_1(M_1)$, such that $\pi_1(M_1)/N$ is virtually a surface group. Algorithms in $\pi_1(M_1)$ can be reduced to similar algorithms in $\pi_1(M_1)/N$, providing solutions (Propositions 5.1 and 5.2). A solution to the conjugacy problem in the Seifert case, already solved in almost all cases by the biautomatic group theory and presenting no difficulty in the (few) remaining cases (namely *NIL* geometry), will only be sketched in Section 5.3.

In the case of a hyperbolic piece M_1 , the biautomatic group theory solves the conjugacy problem. The 2-cosets problem will be solved using the hyperbolic structure of $\pi_1(M_1)$ relative to its boundary

subgroups (in the sense of Farb [9]). To solve the boundary parallelism problem, we will make use of Thurston's hyperbolic surgery theorem to obtain two closed hyperbolic manifolds by Dehn filling on M_1 , and then reducing the problem in $\pi_1(M_1)$ to analogous problems in the groups of these two manifolds. Solutions will then be provided using the word hyperbolic group theory.

1. Reducing the problem

The aim of this section is to reduce the conjugacy problem in the group of a geometrizable (oriented) 3-manifold into the same problem in the more restricted case of a closed irreducible 3-manifold which is either Haken, or a Seifert fibered space, or modelled on *SOL* geometry. That is, if the conjugacy problem is solvable in the group of any such 3-manifold, then it is also solvable in the group of any geometrizable 3-manifold. This result constitutes Lemma 1.4; the reduction will be done in three steps: first reducing to a closed manifold by “doubling the manifold”, then to an irreducible closed manifold by using the Kneser–Milnor decomposition, and finally with the classification theorem for closed irreducible geometrizable 3-manifolds.

1.1. Reducing to the case of a closed manifold

Suppose M is a 3-manifold with a non-empty boundary. Consider a homeomorphic copy M' of M , and a homeomorphism $\varphi : M \rightarrow M'$. Glue M and M' together along the homeomorphisms induced by φ on the boundary components, to obtain a closed 3-manifold, which will be called $2M$. We can reduce the conjugacy problem in $\pi_1(M)$ to the conjugacy problem in $\pi_1(2M)$, by using the following lemma.

Lemma 1.1. $\pi_1(M)$ naturally embeds in $\pi_1(2M)$. Moreover, two elements $u, v \in \pi_1(M)$ are conjugate in $\pi_1(M)$ if and only if they are conjugate in $\pi_1(2M)$.

Proof. For more convenience $2M$ can be seen as $2M = M \cup M'$, with $\partial M = \partial M' = M \cap M'$. There exists a natural homeomorphism $\phi : M \rightarrow M'$ which restricts to identity on ∂M . Consider the natural (continuous) map $\pi : 2M \rightarrow M$ defined by $\pi|_M = Id_M$ and $\pi|_{M'} = \phi^{-1}$.

The inclusion $M \subset 2M$ induces a group homomorphism $\pi_1(M) \rightarrow \pi_1(2M)$. We first need to prove that this map is injective. It suffices to show that any loop in M , contractible in $2M$, is in fact contractible in M . Consider a loop l in M , that is, $l : S^1 \rightarrow M$, and suppose that there exists a map $h : D^2 \rightarrow 2M$ such that h restricted to $S^1 = \partial D^2$ is l ; note $\bar{h} = \pi \circ h$. Since the loop l lies in M , \bar{h} restricted to ∂D^2 is l . Hence l is contractible in M , which proves the first assertion.

We now prove the second assertion. Since $\pi_1(M)$ is a subgroup of $\pi_1(2M)$, the direct implication is obvious. We need to prove the converse. Consider two loops l_u and l_v in M which represent, respectively, u and v . Suppose also that u and v are conjugate in $\pi_1(2M)$. So l_u and l_v are freely homotopic in $2M$. Thus, there exists a map $f : S^1 \times I \rightarrow 2M$, such that f restricted to $S^1 \times 0$ is l_u and f restricted to $S^1 \times 1$ is l_v ; note that $\bar{f} = \pi \circ f$. Since l_u, l_v lie in M , \bar{f} is a homotopy from l_u to l_v in M , and so u and v are conjugate in $\pi_1(M)$. \square

Suppose one needs to solve the conjugacy problem in $\pi_1(M)$ where M is a 3-manifold with a non-empty boundary. By doubling the manifold M along its boundary, one obtains the closed 3-manifold $2M$. If the conjugacy problem in $\pi_1(2M)$ admits a solution, then one can deduce a solution in $\pi_1(M)$. Consider u and v in $\pi_1(M)$. Under the natural embedding $\pi_1(M) \hookrightarrow \pi_1(2M)$, u and v can be seen as elements of

$\pi_1(2M)$. With the preceding lemma, one only needs to check whether u and v are conjugate in $\pi_1(2M)$ to determine whether they are conjugate in $\pi_1(M)$. Hence, the conjugacy problem in $\pi_1(M)$ reduces to the conjugacy problem in $\pi_1(2M)$. Together with the following lemma, the conjugacy problem in geometrizable (oriented) 3-manifolds reduces to the conjugacy problem in closed (oriented) geometrizable 3-manifolds.

Lemma 1.2. *If M is geometrizable, then so is $2M$.*

Proof. A 3-manifold is geometrizable if and only if all of its prime factors are geometrizable. Together with the Kneser–Milnor theorem, if M splits as $M = \#M_i$, then M is geometrizable if and only if each of its (not necessarily prime) factors M_i are geometrizable; this observation will be denoted by $(*)$.

We suppose M to be geometrizable, and wish to show that $2M$ is geometrizable. We denote C_i (resp. D_i) the prime factors of M with an empty (resp. $\neq \emptyset$) boundary; then $M = (\#C_i)\#(\#D_j)$ and $2M = (\#C_i)\#(\#C'_i)\#(\#2D_j)$ where C'_i are homeomorphic copies of C_i . Bear in mind that the $2D_j$ are not necessarily prime. Using $(*)$ it suffices to show that all the $2D_j$ are geometrizable. Hence we will suppose in the following that M is prime with a non-empty boundary; besides, we suppose that $\partial M \not\cong S^2$ because otherwise $M = B^3$ and $2M = S^3$.

If M is not ∂ -irreducible then it must contain an essential disk. If M contains a separating essential disk, $M = M_1\#_{D^2}M_2$, then $2M$ contains a separating essential sphere and splits non-trivially as $2M = 2M_1\#2M_2$. Hence, using $(*)$ and the fact that a 3-manifold is not infinitely decomposable as a non-trivial connected sum, we will suppose that M does not contain any separating essential disk. In particular, $2M$ is prime, and if M contains a non-separating disk then $2M$ would be a sphere bundle over S^1 so that, we moreover suppose M to be ∂ -irreducible. Under these hypotheses $2M$ is Haken, and according to Thurston's geometrization theorem, is geometrizable. \square

1.2. Reducing to the case of an irreducible manifold

Suppose now that M is a closed 3-manifold. According to the Kneser–Milnor theorem (cf. [15]), M admits a unique decomposition as a connected sum of prime manifolds, $M = M_1\#M_2\#\dots\#M_n$, where each M_i is either irreducible, or homeomorphic to $S^2 \times S^1$. Its fundamental group $\pi_1(M)$ decomposes in a free product of the $\pi_1(M_i)$ for $i = 1, 2, \dots, n$, that is, $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) * \dots * \pi_1(M_n)$.

Applying the conjugacy theorem for a free product (cf. [20]), $\pi_1(M)$ has a solvable conjugacy problem, if and only if each of the $\pi_1(M_i)$ has a solvable conjugacy problem. Moreover, $\pi_1(S^2 \times S^1)$ is infinite cyclic and therefore admits a solution to the conjugacy problem. So the conjugacy problem in a (closed, geometrizable) 3-manifold group reduces to the conjugacy problem in all (closed, geometrizable) irreducible 3-manifold groups. Together with the last lemma, one obtains

Lemma 1.3. *The conjugacy problem in groups of oriented (resp. geometrizable) 3-manifolds reduces to the conjugacy problem in groups of closed and irreducible-oriented (resp. geometrizable) 3-manifolds.*

1.3. Reducing to particular 3-manifolds

The final step in this reduction comes from the classification theorem for closed irreducible geometrizable manifolds together with all the already known results on the conjugacy problem in the groups of

such manifolds. The following result constitutes theorem 5.3 of [31] slightly adapted to the oriented case.

Theorem 1.1. *Let M be an irreducible, closed geometrizable (oriented) 3-manifold. Then M satisfies one of the following conditions:*

- (i) M is Haken.
- (ii) M is hyperbolic.
- (iii) M is modelled on SOL geometry. This happens exactly when M is finitely covered by an $S^1 \times S^1$ -bundle over S^1 , with a hyperbolic gluing map. In particular, either M is itself an $S^1 \times S^1$ -bundle over S^1 , or M is the union of two twisted I -bundles over the Klein bottle. In this case, M is Haken.
- (iv) M is modelled on \mathbb{S}^3 , \mathbb{E}^3 , $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, NIL, or on the universal cover of $SL(2, \mathbb{R})$. This happens exactly when M is a Seifert fibered space. In this case M is an S^1 -bundle with base an orbifold O_2 , and if e refers to the Euler number of the bundle, and χ to the Euler characteristic of the base O_2 , then the geometry of M is characterized by e and χ , following the table below:

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$\mathbb{S}^2 \times \mathbb{E}^1$	\mathbb{E}^3	$\mathbb{H}^2 \times \mathbb{E}^1$
$e \neq 0$	\mathbb{S}^3	NIL	$\widetilde{SL}(2, \mathbb{R})$

Note that conditions (ii)–(iv) are disjoint (according to the unicity of the geometry involved for a particular 3-manifold) while condition (i) is not disjoint from conditions (ii)–(iv). For example, obviously a torus bundle M over S^1 is Haken, while M is modelled on SOL when its gluing map is Anosov, or M is a Seifert fibered space (and so satisfies condition (iv)) in the case of a reducible or periodic gluing map.

In the case of a closed hyperbolic manifold M , M is the orbit space of a cocompact action of a discrete subgroup of $PSL(2, \mathbb{C})$ on \mathbb{H}^3 . So, $\pi_1(M)$ is word-hyperbolic (in the sense of Gromov, cf. [13,3]), and admits a (very efficient) solution to the conjugacy problem. So we only need to solve the problem in the remaining cases (i), (iii) and (iv). We have finally obtained the main result of this section:

Lemma 1.4. *The conjugacy problem in groups of oriented geometrizable 3-manifolds reduces to groups of oriented closed 3-manifolds which are either Haken, or a Seifert fibered space, or modelled on SOL geometry.*

In the case of a Seifert fiber space M , its group $\pi_1(M)$ is biautomatic, unless M is modelled on NIL (cf. [24,25]). Hence the biautomatic group theory provides a solution to the conjugacy problem in almost all cases. The remaining cases (those modelled on NIL) can easily be solved by direct methods. We will only sketch a solution in Section 5.3.

In the case of a 3-manifold M modelled on SOL geometry, M is either an $S^1 \times S^1$ -bundle over S^1 or obtained by gluing two twisted I -bundles over \mathbb{KB}_2 along their boundary; in particular, M is Haken. We distinguish the SOL case from the Haken case, because we solve separately what we shall call the *generic Haken case* (a Haken closed manifold which is neither an $S^1 \times S^1$ -bundle over S^1 nor obtained by combining two I -twisted bundles over \mathbb{KB}_2 along their boundary) from the remaining Haken cases (namely an $S^1 \times S^1$ -bundle over S^1 , or two I -twisted bundles over \mathbb{KB}_2 glued along their boundary).

The reason for such a distinction is that in this last non-generic case our general strategy fails (because the JSJ decompositions may not be k -acylindrical for some $k > 0$; see Section 4.3). Nevertheless, a solution can easily be established: either they are Seifert fibered, or the conjugacy problem reduces easily to solving elementary equations in $SL(2, \mathbb{Z})$. We will only sketch a solution in Section 7.

The main part of our work will be devoted to the Haken generic case, which represents the major difficulty, and will be treated in detail. While solutions in the remaining cases are only sketched in Sections 5.3 and 7, the interested reader can find detailed solutions in my Ph.D. thesis [29, Sections 5.5 and 7.1].

2. The group of a Haken closed manifold

In this section we study the fundamental group of a Haken closed manifold M . We see how the JSJ decomposition of M provides a splitting of $\pi_1(M)$ as a fundamental group of a graph of groups.

2.1. JSJ decomposition

Let M be a Haken closed manifold. The JSJ theorem (cf. [17]) asserts that there exists an essential surface W embedded in M (possibly $W = \emptyset$), whose connected components consist of tori, such that if one cuts M along W , one obtains a 3-manifold, whose connected components—called the *pieces*—are either Seifert fibered spaces, or atoroidal (i.e. do not contain an essential torus). Moreover, W is minimal up to ambient isotopy, in the class of surfaces satisfying the above conditions. The manifold M can then be reconstructed by gluing the pieces along their boundary components.

The minimality of W has two consequences which will be essential in our work. First if M_1, M_2 are two Seifert pieces glued along one boundary component (in order to reconstruct M), then the gluing map sends a regular fiber of M_1 onto a loop of M_2 which cannot be homotoped to any regular fiber in M_2 : because otherwise one can extend the fibration on the gluing of M_1 and M_2 , contradicting the minimality of the decomposition. This fact is taken up in the following lemma:

Lemma 2.1. *Suppose M_1 and M_2 are two Seifert pieces in the JSJ decomposition of M , which are glued along one boundary component. Then the gluing map sends a regular fiber of M_1 to a loop which cannot be homotoped in M_2 to a regular fiber.*

The second consequence excludes in almost all cases pieces homeomorphic to a thickened torus. The proof is immediate since gluing one piece N with $S^1 \times S^1 \times I$ along one boundary component does not change the homeomorphism class of N while it increases the number of components of W .

Lemma 2.2. *Suppose M is a Haken closed manifold which is not homeomorphic to an $S^1 \times S^1$ -bundle over S^1 . Then, none of the pieces of the JSJ decomposition of M is homeomorphic to $S^1 \times S^1 \times I$.*

We now introduce some notations. Suppose W is non-empty, and admits as connected components $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_q$. Since W is two-sided in M , each of the \mathcal{T}_i admits a regular neighborhood in M homeomorphic to a thickened torus, which we shall denote as $N(\mathcal{T}_i)$, chosen in such a way that all the $N(\mathcal{T}_i)$ (for $i = 1, 2, \dots, q$) are disjoint one to the other. Now when we say “cutting M along W ”

we mean considering the compact 3-manifold $\sigma_W(M)$ defined as

$$\sigma_W(M) = M - \bigcup_i \text{int}(N(\mathcal{T}_i)).$$

The homeomorphism class of $\sigma_W(M)$ does not depend on the neighborhoods involved. The connected components of M have a non-empty boundary when $W \neq \emptyset$. We shall call them the *pieces* of the decomposition, and name them as M_1, M_2, \dots, M_n .

According to Thurston's geometrization theorem (cf. [39]), the atoroidal pieces admit a hyperbolic structure with finite volume. We shall call them the *hyperbolic pieces*.

There exists a canonical map associated with the JSJ decomposition called the *identification map*, $r : \sigma_W(M) \rightarrow M$, which is such that r restricted to $\text{int}(\sigma_W(M))$ is a homeomorphism, and each preimage $r^{-1}(\mathcal{T}_i)$ of \mathcal{T}_i consists of two homeomorphic copies of $S^1 \times S^1$ in $\partial\sigma_W(M)$, which we shall (arbitrarily) call \mathcal{T}_i^- and \mathcal{T}_i^+ .

Then M can be reconstructed from $\sigma_W(M)$. There exists two homeomorphisms:

$$\begin{aligned} \rho_i^- : S^1 \times S^1 &\rightarrow \mathcal{T}_i^- \\ \rho_i^+ : S^1 \times S^1 &\rightarrow \mathcal{T}_i^+ \end{aligned}$$

such that the following diagram commutes:

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\rho_i^-} & \mathcal{T}_i^- \\ \rho_i^+ \downarrow & & \downarrow r \\ \mathcal{T}_i^+ & \xrightarrow{r} & \mathcal{T}_i \end{array}$$

Define $\rho_i : \mathcal{T}_i^- \rightarrow \mathcal{T}_i^+$ by $\rho_i = \rho_i^+ \circ (\rho_i^-)^{-1}$, which will be called *the gluing map associated to the torus \mathcal{T}_i* . Then M is homeomorphic to the manifold obtained by gluing $\sigma_W(M)$ on its boundary, according to the gluing maps $\rho_1, \rho_2, \dots, \rho_q$.

2.2. Splitting the fundamental group

The JSJ decomposition of M provides a splitting of $\pi_1(M)$ as a fundamental group of a graph of groups. The information needed to define the graph of groups comes directly from the information characterizing the JSJ decomposition of M , namely the pieces obtained in this decomposition, and the associated gluing maps.

A JSJ decomposition $W = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \dots \cup \mathcal{T}_q$ of M , and the pieces M_1, M_2, \dots, M_n obtained in this decomposition naturally provide a splitting of M as a graph of space (\mathcal{M}, X) (cf. [32]). The underlying graph X has one vertex $v(M_i)$ (which shall also be denoted as v_i) for each piece M_i ; given a vertex v of X we shall denote the associated piece by $M(v)$. To each connected component \mathcal{T}_j of W correspond two edges of X , $e(\mathcal{T}_j)$ and $e(\mathcal{T}_j)^{-1}$ inverse one to the other (we shall also denote, respectively, e_j and \bar{e}_j), and if M_k, M_l are such that $\mathcal{T}_j^- \subset \partial M_k$ and $\mathcal{T}_j^+ \subset \partial M_l$, then $e(\mathcal{T}_j)$ has origin $\mathbf{o}(e(\mathcal{T}_j)) = v(M_k)$ and extremity $\mathbf{e}(e(\mathcal{T}_j)) = v(M_l)$ (cf. Fig. 1). The gluing maps associated to the edge $e(\mathcal{T}_j)$ are given, respectively, by ρ_i^- and ρ_i^+ . One naturally obtains the graph of groups (\mathcal{G}, X) by assigning to each vertex $v_i = v(M_i)$ the group $\pi_1(M_i, *_i)$ for some $*_i \in M_i$ —which we shall denote by $G(v_i)$ and call a *vertex*

Fig. 1. Construction of the graph X .

group—and to each edge e_j the group $G(e_j)$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ —called an *edge group*. For each edge e_i , say with $\mathbf{o}(e_i) = v_0$ and $\mathbf{e}(e_i) = v_1$, fix one base point $\ast_i^- \in \mathcal{T}_i^-$ (resp. $\ast_i^+ \in \mathcal{T}_i^+$) such that $r(\ast_i^-) = r(\ast_i^+) = \ast_i \in \mathcal{T}_i$ and a path from \ast_0 to \ast_i^- in $M(v_0)$ (resp. from \ast_1 to \ast_i^+ in $M(v_1)$); it defines two monomorphisms $\varphi_j^- : G(e_j) \hookrightarrow G(v_0)$ and $\varphi_j^+ : G(e_j) \hookrightarrow G(v_1)$ by $\varphi_j^- = (\rho_j^-)_*$, $\varphi_j^+ = (\rho_j^+)_*$, which shall also be denoted by $\varphi_{e_j}^-, \varphi_{e_j}^+$. We denote by $G(e_j)^-$ and $G(e_j)^+$ their respective images in the vertex groups and $\varphi_j : G(e_j)^- \rightarrow G(e_j)^+$ the isomorphism given by $\varphi_j = \varphi_j^+ \circ (\varphi_j^-)^{-1}$. Now for an arbitrary edge e , $G(\bar{e}) = G(e)$, $G(\bar{e})^- = G(e)^+$, $G(\bar{e})^+ = G(e)^-$, $\varphi_{\bar{e}}^- = \varphi_e^+$, $\varphi_{\bar{e}}^+ = \varphi_e^-$, and $\varphi_{\bar{e}} = \varphi_e^{-1}$.

Once a base point $\ast \in X$ is given, one defines the fundamental group $\pi_1(\mathcal{G}, X, \ast)$ of the pointed graph of groups (\mathcal{G}, X, \ast) ; it turns out that this definition does not depend on $\ast \in X$ (cf. [36]) and that $\pi_1(\mathcal{G}, X)$ is naturally isomorphic to $\pi_1(M)$ (cf. [32]). The vertex groups naturally embed in $\pi_1(\mathcal{G}, X)$ while edge groups embed in vertex groups; their respective images will be called *vertex subgroups* of $\pi_1(M)$ and *edge subgroups* of vertex groups (hence of $\pi_1(M)$).

The splitting of $\pi_1(M)$ as a graph of groups provides a (finite) presentation, once a maximal tree T in X is chosen. An edge of X will be said to be *T-separating* if it belongs to T , and *T-non-separating* otherwise. If for each $i = 1, 2, \dots, n$ the vertex group $\pi_1(M_i)$ admits the (finite) presentation $\langle S_i \mid R_i \rangle$ then $\pi_1(M)$ admits the following presentation:

Generators: $S_1 \cup S_2 \cup \dots \cup S_n \cup \{t_e \mid e \text{ is an edge of } X\}$.

Relators:

$$\begin{aligned} R_1 \cup R_2 \cup \dots \cup R_n \cup \{ & \text{for all edge } e \text{ of } X, \text{ for all } c \in G(e)^-, \quad t_e \varphi_e(c) t_e^{-1} = c \} \\ & \cup \{ \text{for all edge } e \text{ of } X, \quad t_{\bar{e}} = t_e^{-1} \} \\ & \cup \{ \text{for all } T\text{-separating edge } e, \quad t_e = 1 \}. \end{aligned}$$

A generator t_e for some edge e will be called a *stable letter* associated to the edge e ; for an arbitrary edge e_i , we may also denote t_i instead of t_{e_i} . Note that since the edge groups are finitely generated, one immediately obtains a finite presentation for $\pi_1(M)$ by replacing the relations $\forall c \in G(e)^-, \quad t_e c t_e^{-1} = \varphi_e(c)$, by the same relations involving two generators c_1, c_2 of $G(e)^-$.

2.3. Algorithmically splitting the group

Suppose the closed Haken manifold M is given, in some manner, for example by a triangulation, a Heegard splitting, or a Dehn filling on a link. There exists an algorithm (cf. [18]), which provides a JSJ decomposition of M (as well as the associated gluing maps). This algorithm uses an improvement of the theory of normal surfaces due to Haken (cf. [14]). It searches until it finds a maximal system of essential tori, which provides a JSJ decomposition of M . Moreover, this algorithm finds Seifert invariants associated to each Seifert piece.

Once the JSJ decomposition, the pieces and the gluing maps are given, one can easily split $\pi_1(M)$ as a group of a graph of groups, as described above. When we will be algorithmically working on $\pi_1(M)$, we will suppose a canonical presentation of $\pi_1(M)$ is given, that is, a presentation of $\pi_1(M)$ as above, such that the Seifert pieces are given with their canonical presentation (in such a way that they can be identified as being Seifert pieces, and implicitly provide Seifert fibrations, cf. Section 4.1).

3. The conjugacy theorem

As seen before, the JSJ decomposition of a Haken closed manifold M provides a splitting of $\pi_1(M)$ as a fundamental group of a graph of groups $\pi_1(M) = \pi_1(\mathcal{G}, X)$. This fact will allow to establish a conjugacy theorem (i.e. which characterizes conjugate elements) in $\pi_1(M)$ in the spirit of analogous results in amalgams or HNN extensions (cf. [20,19]). This is the aim of this section. We will first need to recall classical ways to write down an element of a group of a graph of groups in a “reduced form” before stating the main result, that is, Theorem 3.1.

3.1. Cyclically reduced form

We first need to give some definitions.

We note a path in X in the extended manner: $(v_{\sigma_1}, e_{\tau_1}, v_{\sigma_2}, e_{\tau_2}, \dots, v_{\sigma_m}, e_{\tau_m}, v_{\sigma_{m+1}})$, where the v_{σ_i} and the e_{τ_i} are, respectively, vertices and edges of X such that $\mathbf{e}(e_{\tau_i}) = v_{\sigma_{i+1}}$ and $\mathbf{o}(e_{\tau_i}) = v_{\sigma_i}$ for $i = 1, \dots, m$. The vertices v_{σ_1} and $v_{\sigma_{m+1}}$ are its two endpoints; a loop is a path where two endpoints coincide.

Given an arbitrary graph of group (\mathcal{G}, X) , we recall that in Bass–Serre’s terminology [36] a *word of type* \mathcal{C} is a couple (\mathcal{C}, μ) where:

- \mathcal{C} is a based loop in X , say $\mathcal{C} = (v_{\sigma_1}, e_{\tau_1}, v_{\sigma_2}, e_{\tau_2}, \dots, v_{\sigma_m}, e_{\tau_m}, v_{\sigma_{m+1}})$.
- μ is a sequence $\mu = (\mu_1, \mu_2, \dots, \mu_{m+1})$, such that $\forall i = 1, \dots, m+1, \mu_i \in G(v_{\sigma_i})$; μ_i will be called the *label* of the vertex v_{σ_i} .

The *length* of a word of type \mathcal{C} is defined to be the length of the loop \mathcal{C} . Once a base point $*$ in X is given, each word of type \mathcal{C} for some loop \mathcal{C} with base point $*$ defines an element of $\pi_1(\mathcal{G}, X, *)$ (cf. [36]) which we will call its *label*; the label of (\mathcal{C}, μ) will be denoted $|\mathcal{C}, \mu|$ and we shall discuss the *form* (\mathcal{C}, μ) for $|\mathcal{C}, \mu|$. When one considers a presentation as in Section 2.2, $|\mathcal{C}, \mu| = \mu_1.t_{\tau_1}.\mu_2.t_{\tau_2} \cdots \mu_m.t_{\tau_m}.\mu_{m+1}$.

A word of type \mathcal{C} , (\mathcal{C}, μ) is said to be *reduced* if either its length is 0 and its label is $\neq 1$, or its length is greater than 1 and each time $e_{\tau_{i-1}} = \bar{e}_{\tau_i}$ then $\mu_i \in G(v_{\tau_i}) \setminus G(e_{\tau_i})^+$. We shall discuss of a *reduced form* for

its label $|\mathcal{C}, \mu|$. Now given any non-trivial element $g \in \pi_1(\mathcal{G}, X)$ there exists a reduced form associated with g (cf. [36]).

Definitions. A *cyclic conjugate* of $(\mathcal{C}, \mu) = ((v_{\sigma_1}, e_{\tau_1}, v_{\sigma_2}, \dots, e_{\tau_n}, v_{\sigma_{n+1}}), (\mu_1, \mu_2, \dots, \mu_n, \mu_{n+1}))$ is $(\mathcal{C}', \mu') = ((v_{\sigma_i}, e_{\tau_i}, v_{\sigma_{i+1}}, \dots, e_{\tau_n}, v_{\sigma_{n+1}}, e_{\tau_1}, \dots, e_{\tau_{i-1}}, v_{\sigma_i}), (\mu_i, \mu_{i+1}, \dots, \mu_n, \mu_{n+1}\mu_1, \dots, \mu_{i-1}, 1))$, for some $i, 0 \leq i \leq n$, (indices are taken modulo n).

- A word of type \mathcal{C} is a *cyclically reduced form*, if all of its cyclic conjugates are reduced, and if $\mu_{n+1} = 1$ when $n > 1$ (hence this last vertex becomes superfluous and should be ignored).

One can associate to any non-trivial conjugacy class in $\pi_1(\mathcal{G}, X)$ a cyclically reduced form whose label is an element of the class. Just start from an element $g \neq 1$ and find a reduced form for g ; if it is not cyclically reduced then reduce one of its cyclic conjugates. Apply this process as long as it is possible; since the length of the form strictly decreases it must stop providing a cyclically reduced form whose label is a conjugate of g .

The crucial property for reduced forms is that given two reduced forms (\mathcal{C}, μ) and (\mathcal{C}', μ') such that $|\mathcal{C}, \mu| = |\mathcal{C}', \mu'|$, then necessarily $\mathcal{C} = \mathcal{C}' = (v_{\sigma_1}, e_{\tau_1}, v_{\sigma_2}, \dots, e_{\tau_n}, v_{\sigma_{n+1}})$ and there exists a sequence $(c_1, c_2, \dots, c_n), c_i \in G(e_{\tau_i})$ such that $\mu_1 = \mu'_1 \cdot (c_1^-)^{-1}$, $\mu_{n+1} = c_n^+ \cdot \mu'_{n+1}$ in $G(v_{\sigma_1})$ and $\mu_i = c_{i-1}^+ \cdot \mu'_i \cdot (c_i^-)^{-1}$ in $G(v_{\sigma_i})$ for $i = 2, 3, \dots, n$ (cf. Section 5.2, [36]). The conjugacy theorem in the next section gives an analogue of this property when one considers cyclically reduced forms and conjugacy classes instead of reduced forms and elements.

3.2. The conjugacy theorem

In this section, we will consider a graph of groups (\mathcal{G}, X) associated to the JSJ decomposition of an arbitrary Haken closed manifold M . We prove in this case the conjugacy theorem in the group of such a graph of groups. Nevertheless, the theorem remains true for any graph of groups: we prove this result using a different method and in full generality in a work in preparation.

Theorem 3.1. Suppose (\mathcal{C}, μ) and (\mathcal{C}', μ') are cyclically reduced forms, whose labels ω and ω' are conjugate in $\pi_1(M)$. Then, (\mathcal{C}, μ) and (\mathcal{C}', μ') have the same length, and moreover, either:

- their length is equal to 0, $\mathcal{C} = \mathcal{C}' = (v_{\sigma_1})$, and ω, ω' are conjugate in $G(v_{\sigma_1})$;
- their length is equal to 0, and there exists a path $(v_{\alpha_0}, e_{\beta_1}, \dots, e_{\beta_p}, v_{\alpha_p})$ in X , and a sequence (c_1, c_2, \dots, c_p) with $\forall i = 1, 2, \dots, p$, c_i lying in the edge group $G(e_{\beta_i})$, such that $\omega \in G(v_{\alpha_0})$, $\omega' \in G(v_{\alpha_p})$, and

$$\omega \sim c_1^- \quad \text{in } G(v_{\alpha_0})$$

$$\omega' \sim c_p^+ \quad \text{in } G(v_{\alpha_p})$$

$$\text{and } \forall i = 1, 2, \dots, p-1, \quad c_i^+ \sim c_{i+1}^- \quad \text{in } G(v_{\alpha_i}), \text{ or}$$

- their length is greater than 0. Up to the cyclic permutation of (\mathcal{C}', μ') , the loops $\mathcal{C}, \mathcal{C}'$ are equal, $\mathcal{C} = \mathcal{C}' = (v_{\sigma_1}, e_{\tau_1}, \dots, v_{\sigma_n}, e_{\tau_n})$, and there exists a sequence (c_1, \dots, c_n) , with for all $i = 1, \dots, n$,

c_i lying in the edge group $G(e_{\tau_i})$, such that:

$$\begin{aligned} \mu_1 &= c_n^+ \cdot \mu'_1 \cdot (c_1^-)^{-1} \quad \text{in } G(v_{\sigma_1}) \\ \forall i = 2, 3, \dots, n \quad \mu_i &= c_{i-1}^+ \cdot \mu'_i \cdot (c_i^-)^{-1} \quad \text{in } G(v_{\sigma_i}) \end{aligned}$$

in particular, the element $c_n^+ \in G(e_{\tau_n})^+$ conjugates ω' into ω in $\pi_1(M)$:

$$\omega = c_n^+ \cdot \omega' \cdot (c_n^+)^{-1} \quad \text{in } \pi_1(M).$$

(Recall that if c lies in the edge group $G(e_i)$, we note $c^- = \varphi_i^-(c) \in G(e_i)^-$ and $c^+ = \varphi_i^+(c) \in G(e_i)^+$.)

Proof. Consider two cyclically reduced forms (\mathcal{C}, μ) and (\mathcal{C}', μ') with respective labels ω and ω' .

In order to define edge groups, vertex subgroups and the embeddings we have considered in Section 2.2 one base point $*_i$ in each piece M_i , one base point $\underline{*}_j$ in each torus component of W , and for each edge e_j with $\mathbf{o}(e_j) = v_k$, $\mathbf{e}(e_j) = v_l$ two paths that we denote $[*_k, \underline{*}_j^-]$ and $[*_l, \underline{*}_j^+]$, respectively, in M_k from $*_k$ to $\underline{*}_j^-$ and in M_l from $*_l$ to $\underline{*}_j^+$. We deform by homotopy keeping their endpoints fixed, all of these paths in such a way that once we have noted $[e_j] = [*_k, \underline{*}_j^-] \cdot [*_l, \underline{*}_j^+]^{-1}$, all such $[e_j]$ become smooth and transverse with W in M .

One constructs a smooth based-loop in M representing (\mathcal{C}, μ) in the following way: suppose $\mathcal{C} = (v_{\sigma_1}, e_{\tau_1}, v_{\sigma_2}, e_{\tau_2}, \dots, e_{\tau_n})$; for each vertex v_{σ_i} of \mathcal{C} choose a smooth loop V_i with base point $*_{\sigma_i}$ in $\text{int}(M_{\sigma_i}) \subset M$, which represents the label μ_i of v_{σ_i} in $\pi_1(M_{\sigma_i}, *_{\sigma_i})$. Replace in \mathcal{C} the vertex v_{σ_i} by this loop. Replace each edge e_{τ_j} in \mathcal{C} by the path $[e_{\tau_j}]$ of M . Finally, concatenate the elements of the sequence obtained and deform by small $*_{\sigma_i}$ -homotopy each V_i to obtain a smooth path \mathcal{P}_ω , with base point $*_{\sigma_1}$. Proceed in the same way to obtain a smooth based-path $\mathcal{P}_{\omega'}$ representing (\mathcal{C}', μ') .

Suppose that ω and ω' are conjugate in $\pi_1(M)$. Then the loops \mathcal{P}_ω and $\mathcal{P}_{\omega'}$ are freely homotopic in M . Hence, there exists a map $f : S^1 \times I \rightarrow M$, such that f restricted to $S^1 \times 0$ is \mathcal{P}_ω , and f restricted to $S^1 \times 1$ is $\mathcal{P}_{\omega'}$. One can also suppose that f is smooth. Since \mathcal{P}_ω and $\mathcal{P}_{\omega'}$ are transverse to W , according to the homotopy transversality theorem one can deform f without changing either \mathcal{P}_ω or $\mathcal{P}_{\omega'}$, such that it becomes transverse to W . Then, by the transversality theorem, $f^{-1}(W)$ is a compact 1-submanifold of $S^1 \times I$, such that $\partial(f^{-1}(W)) = f^{-1}(W) \cap \partial(S^1 \times I)$; hence, $f^{-1}(W)$ consists of disjoint segments and circles properly embedded in $S^1 \times I$. Among all the ways to choose and deform f as above, we consider one such that the map f obtained after deforming is minimal, in the sense that the number of connected components of $f^{-1}(W)$ is minimal.

The minimality in the choice of f implies that none of the circle components of $f^{-1}(W)$ bound a disk, while the fact that (\mathcal{C}, μ) and (\mathcal{C}', μ') are cyclically reduced forms implies that none of the segment components of $f^{-1}(W)$ have its two boundary components both in $S^1 \times 0$ or in $S^1 \times 1$. Hence if $f^{-1}(W)$ is non-empty, then it consists either of disjoint circles parallel to the boundary or of disjoint segments joining $S^1 \times 0$ to $S^1 \times 1$ (cf. Fig. 2).

First we can conclude that \mathcal{P}_ω and $\mathcal{P}_{\omega'}$ intersect W the same number of times. So the cyclically reduced forms (\mathcal{C}, μ) and (\mathcal{C}', μ') must have the same length.

Case 1: Suppose that $f^{-1}(W)$ is empty. So the annulus $f(S^1 \times I)$ lies in some piece, say $\text{int}(M_i)$. It implies that (\mathcal{C}, μ) , (\mathcal{C}', μ') both have length 0, and moreover that ω, ω' belong to $\pi_1(M_i)$ and are conjugate in $\pi_1(M_i)$; hence, conclusion (i) holds.

Fig. 2.

Case 2: Suppose $f^{-1}(W)$ consists of p circles, C_1, C_2, \dots, C_p . In this case (\mathcal{C}, μ) and (\mathcal{C}', μ') both have length equal to 0. We denote $C_0 = S^1 \times 0$, $C_{p+1} = S^1 \times 1$, and consider the circles C_i as based loops such that they are all isotopic in $S^1 \times I$ and $f \circ C_0 = \mathcal{P}_\omega$, $f \circ C_{p+1} = \mathcal{P}_{\omega'}$. We will proceed by induction on p to show that conclusion (ii) holds.

Consider first the case $p = 1$. If one cuts $S^1 \times I$ along C_1 it decomposes into two annuli bounded, respectively, by C_0, C_1 and C_1, C_2 which map, respectively, under f , say in M_k, M_l ; denote \mathcal{T}_i the component of W in which C_i maps and $e_i = e(\mathcal{T}_i)^{\pm 1}$ the edge such that $\mathbf{o}(e_i) = v_k$, $\mathbf{e}(e_i) = v_l$. The loop $f \circ C_0 = \mathcal{P}_\omega$ (resp. $f \circ C_2 = \mathcal{P}_{\omega'}$) represents the element ω in $G(v_k)$ (resp. ω' in $G(v_l)$). The loop $f \circ C_1 \subset \mathcal{T}_i$ defines a conjugacy class $[c_1]$ in $G(e_i)$ such that $\omega \sim c_1^-$ in $G(v_k)$ and $\omega' \sim c_1^+$ in $G(v_l)$. Hence conclusion (ii) holds when one considers the path (v_k, e_i, v_l) and the sequence (c_1) .

Consider now the case $p > 1$. Suppose that conclusion (ii) holds whenever $f^{-1}(W)$ consists of $p - 1$ circles, and moreover that $f^{-1}(W)$ has p components. The loops C_{p-1} and C_p cobound an annulus A which maps in, say M_k ; denote \mathcal{T}_i the component of W in which C_p maps. Consider an additional loop C in $\text{int}(A)$ isotopic in A with both C_{p-1} and C_p . Then $f \circ C$ defines a conjugacy class $[\omega_0]$ in $G(v_k)$, and once $e_i = e(\mathcal{T}_i)^{\pm 1}$ is judiciously chosen, $f \circ C_p$ defines a conjugacy class $[c_p]$ in $G(e_i)$ such that $c_p^- \sim \omega_0$ in $G(v_k)$. Then cut $S^1 \times I$ along C : it decomposes into two parts: the former one A_0 containing $S^1 \times 0$ and the latter A_1 containing $S^1 \times 1$. The hypothesis of induction can be applied when one restricts f to the annulus A_0 to provide a path, say $\mathcal{P} = (v_0, e_1, \dots, e_{p-1}, v_k)$ with endpoint v_k and a sequence $c = (c_1, \dots, c_{p-1})$ as in conclusion (ii) from ω to ω_0 . The same argument as in the former case $p = 1$ applied to the annulus A_1 provides the path $(v_k, e_i, \mathbf{e}(e_i))$ and the sequence (c_p) from ω_0 to ω' . Then conclusion (ii) holds when one considers the path $\mathcal{P} \cdot (v_k, e_i, \mathbf{e}(e_i))$ together with the sequence $c \cdot (c_p)$ (Fig. 3).

Case 3: Suppose $f^{-1}(W)$ consists of $n > 0$ segments; then, n is the length of both (\mathcal{C}, μ) and (\mathcal{C}', μ') . We must show that conclusion (iii) holds. We note that $\mathcal{C} = (v_{\sigma_1}, e_{\tau_1}, v_{\sigma_2}, e_{\tau_2}, \dots, e_{\tau_n})$, while \mathcal{C}' is clearly a cyclic conjugate of \mathcal{C} . For more convenience during the rest of the proof indices will be assigned modulo n . We denote x_1, x_2, \dots, x_n the points of $\mathcal{P}_\omega^{-1}(W)$ in such a way that if one starts from $(1, 0)$ and turns in the positive sense on $S^1 \times 0$, one encounters x_1, x_2, \dots, x_n in this order, and we proceed the same way with the points x'_1, x'_2, \dots, x'_n of $\mathcal{P}_{\omega'}^{-1}(W)$; they decompose $S^1 \times 0$ and $S^1 \times 1$ into paths which will be, respectively, denoted $[x_{i-1}, x_i]$ and $[x'_{i-1}, x'_i]$, $i = 1, 2, \dots, n$. Consider the segments of $f^{-1}(W)$ as paths C_1, C_2, \dots, C_n such that each C_i starts in $x_i \in S^1 \times 0$ and ends in some $x'_j \in S^1 \times 1$. Necessarily there exists an integer p such that $\forall i = 1, \dots, n$, the path C_i ends in $x'_j \in S^1 \times 1$ with $j = i + p$. By changing if necessary (\mathcal{C}', μ') into a cyclic conjugate, we can suppose that $p = 0$; hence, with this convention $\mathcal{C} = \mathcal{C}'$ and the paths C_i go from x_i to x'_i . The annulus decomposes into n strips such that the boundary of strip i contains the loop $[x_{i-1}, x_i] \cdot C_i \cdot [x'_i, x'_{i-1}]^{-1} \cdot C_{i-1}^{-1}$.

Fig. 3.

By construction each of the C_i maps under f on a loop in \mathcal{T}_{τ_i} with base point $f(x_i) = f(x'_i) = \underline{x}_{\tau_i}$. Hence, $f \circ C_i$ defines an element $c_i \in G(e_{\tau_i})$. Consider for $i = 1, 2, \dots, n$, the loops C_i^- and C_i^+ defined by

$$\begin{aligned} C_i^- &= [*_{\sigma_i}, \underline{x}_{\tau_i}^-].f \circ C_i.[\underline{x}_{\tau_i}^-, *_{\sigma_i}], \\ C_i^+ &= [*_{\sigma_{i+1}}, \underline{x}_{\tau_i}^+].f \circ C_i.[\underline{x}_{\tau_i}^+, *_{\sigma_{i+1}}]. \end{aligned}$$

The loop C_i^- (resp. C_i^+) has base point $*_{\sigma_i}$ (resp. $*_{\sigma_{i+1}}$) and represents the element $c_i^- \in G_{\tau_i}^- \subset \pi_1(M_{\sigma_i})$ (resp. $c_i^+ \in G_{\tau_i}^+ \subset \pi_1(M_{\sigma_{i+1}})$), with $c_i^+ = \varphi_i^+(c_i)$ et $c_i^- = \varphi_i^-(c_i)$.

Consider also for $i = 1, 2, \dots, n$ the loops W_i and W'_i in $\text{int}(M_{\sigma_i})$, with base point $*_{\sigma_i}$ defined by (Fig. 4):

$$\begin{aligned} W_i &= [*_{\sigma_i}, \underline{x}_{\tau_{i-1}}^+].f \circ [x_{i-1}, x_i].[\underline{x}_{\tau_i}^-, *_{\sigma_i}], \\ W'_i &= [*_{\sigma_i}, \underline{x}_{\tau_{i-1}}^+].f \circ [x'_{i-1}, x'_i].[\underline{x}_{\tau_i}^-, *_{\sigma_i}]. \end{aligned}$$

By construction, once we denote $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ and $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_n)$, then the loops W_i and W'_i represent, respectively, the elements μ_i and μ'_i of $\pi_1(M_{\sigma_i})$.

The strips show that for $i = 1, 2, \dots, n$ the paths $f \circ [x_{i-1}, x_i]$ and $f \circ C_{i-1}.f \circ [x'_{i-1}, x'_i].f \circ C_i^{-1}$ are homotopic in M_{σ_i} with endpoints $\underline{x}_{\tau_{i-1}}, \underline{x}_{\tau_i}$ fixed. Hence for all $i = 1, 2, \dots, n$, one has in M_{σ_i} the $*_{\sigma_i}$ -homotopy:

$$\begin{aligned} C_{i-1}^+.W'_i.(C_i^-)^{-1} &\approx_{*_{\sigma_i}} [*_{\sigma_i}, \underline{x}_{\tau_{i-1}}^+].f \circ C_{i-1}.f \circ [x'_{i-1}, x'_i].f \circ C_i^{-1}.[\underline{x}_{\tau_i}^-, *_{\sigma_i}] \\ &\approx_{*_{\sigma_i}} [*_{\sigma_i}, \underline{x}_{\tau_{i-1}}^+].f \circ [x_{i-1}, x_i].[\underline{x}_{\tau_i}^-, *_{\sigma_i}] \\ &\approx_{*_{\sigma_i}} W_i \end{aligned}$$

and one can slightly deform the paths on regular neighborhoods of $\mathcal{T}_{\tau_{i-1}}$ and \mathcal{T}_{τ_i} such that the homotopy takes place in $\text{int}(M_{\sigma_i})$. Hence, for all $i = 1, 2, \dots, n$, $\mu_i = c_{i-1}^+.\mu'_i.(c_i^-)^{-1}$ in $\pi_1(M_{\sigma_i})$.

Fig. 4.

Now,

$$\begin{aligned}\omega &= \mu_1.t_{\tau_1}.\mu_2.t_{\tau_2} \cdots t_{\tau_{n-1}}.\mu_n.t_{\tau_n}, \\ \omega &= c_n^+.\mu'_1.(c_1^-)^{-1}.t_{\tau_1}.c_1^+.\mu'_2.(c_2^-)^{-1}.t_{\tau_2}.c_2^+ \cdots (c_{n-1}^-)^{-1}.t_{\tau_{n-1}}.c_{n-1}^+.\mu'_n.(c_n^-)^{-1}.t_{\tau_n}, \\ \omega &= c_n^+.\mu'_1.t_{\tau_1}.\mu'_2.t_{\tau_2} \cdots t_{\tau_{n-1}}.\mu'_n.t_{\tau_n}.(c_n^+)^{-1}\end{aligned}$$

shows that $\omega = c_n^+.\omega'.(c_n^+)^{-1}$ in $\pi_1(M)$, with $c_n^+ \in G(e_{\tau_n})^+$. So in this case conclusion (iii) holds, which concludes the proof. \square

4. Reducing conjugacy problem in $\pi_1(M)$ to problems in the pieces

In this section we establish the main argument for solving the conjugacy problem in the group of a Haken closed manifold M (which is not an $S^1 \times S^1$ -bundle or two twisted I -bundles over \mathbb{KB}_2 glued along their boundary). We reduce the conjugacy problem to three elementary problems in the group of the pieces obtained in a non-trivial JSJ decomposition of M : namely the conjugacy problem (of course), the boundary parallelism problem and the 2-cosets problem.

The *boundary parallelism problem* consists in, given a boundary subgroup T of the group $\pi_1(N)$ of a piece N , to decide for any element ω of $\pi_1(N)$ (given as words on a given set of generators), whether ω is conjugate in $\pi_1(N)$ to an element of T .

The *2-cosets problem* consists in, given two boundary subgroups N_1, N_2 of $\pi_1(N)$ (possibly identical), to find for any $u, v \in \pi_1(N)$, all the couples $(c_1, c_2) \in N_1 \times N_2$ which are solutions of the equation $u = c_1.v.c_2$ in $\pi_1(N)$.

We show that if one can solve those three problems in the groups of the pieces obtained, then one can solve the conjugacy problem in the group of M (Theorem 4.1). Note first that we will suppose that a canonical presentation of $\pi_1(M)$, that is, its decomposition as a graph of groups, as well as canonical presentations for the groups of the Seifert pieces, are given. Indeed, given the manifold M , there exists

an algorithm based along the same lines as the Haken theory of normal surfaces, which provides a minimal JSJ decomposition of M , as well as fibrations of the Seifert pieces [18]. Moreover, given a finite presentation of the group of a Haken manifold M , one can reconstruct a triangulation of the manifold M .

In the previous section we established a theorem characterizing conjugate elements, which does not directly provide a solution to the conjugacy problem, but which is essential to the reduction. On its own, this result does not allow such a reduction, but the key point is that groups of the pieces of a JSJ decomposition have algebraic properties (a kind of “malnormality” for the boundary subgroups, Proposition 4.1) which together with Lemma 2.1 (a consequence of the minimality of the JSJ decomposition), make the reduction process work. These algebraic properties will be established in Section 4.2, and will imply the k -acylindricity of the JSJ splitting (Section 4.3), as well as the existence of an algorithm to write words in cyclically reduced forms (Section 4.4), which are all essential to the reduction process (Section 4.5). But first, we recall some elementary facts upon Seifert fiber spaces (we refer the reader to [33,17,16,27]).

4.1. Reviews on Seifert fiber spaces

Let M be a Seifert fibered space. A Seifert fibration of M is characterized by a set of invariants (up to fiber-preserving homeomorphism) of one of the forms:

$$(o, g, p, b \mid \alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_q, \beta_q),$$

$$(n, g, p, b \mid \alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_q, \beta_q).$$

The former case occurs when the base is oriented (“ o ” stands for “oriented”), and the latter when the base is non-oriented (“ n ” for “non-oriented”). The numbers “ g ”, “ p ” are, respectively, the genus of the base, and the number of its boundary components. The number b is related to the Euler number of the S^1 -bundle associated to the fibration, and q is the number of exceptional fibers; α_i is the index of the i th exceptional fiber, $0 < \beta_i < \alpha_i$. This exceptional fiber has type (α_i, β_i) .

A Seifert fiber space may admit several fibrations, but they remain isolated cases: with the exception of lens spaces, prism manifolds, a solid torus, a twisted I -bundle over $\mathbb{K}\mathbb{B}_2$, or the double of a twisted I -bundle over $\mathbb{K}\mathbb{B}_2$, a Seifert fiber spaces can be endowed with a unique Seifert fibration [16, Theorem VI-17].

Now, given a set of invariants of its Seifert fibration, $\pi_1(M)$ admits a canonical presentation, of one of the forms, according to whether its base is oriented or non-oriented (cf. [16]).

$$\left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_q, d_1, \dots, d_p, h \mid \right.$$

$$[a_i, h] = [b_i, h] = [c_j, h] = [d_k, h] = 1; c_j^{\alpha_j} = h^{\beta_j}; h^b = \left(\prod_{i=1}^g [a_i, b_i] \right) c_1 \cdots c_q d_1 \cdots d_p \Bigg\rangle \quad (1)$$

$$\left\langle a_1, \dots, a_g, c_1, \dots, c_q, d_1, \dots, d_p, h \mid \right.$$

$$a_i h a_i^{-1} = h^{-1}; [c_j, h] = [d_k, h] = 1; c_j^{\alpha_j} = h^{\beta_j}; h^b = \left(\prod_{i=1}^g a_i^2 \right) c_1 \cdots c_q d_1 \cdots d_p \Bigg\rangle \quad (2)$$

with $1 \leq i \leq g$, $1 \leq j \leq q$, and $1 \leq k \leq p$. The generator h is the class of a (any) regular fiber, and if \mathcal{T}_k is a component of ∂M , the associated boundary subgroup T_k is generated by h, d_k . The element h generates a normal subgroup $N = \langle h \rangle$ of $\pi_1(M)$, called the *fiber*. Moreover, if $\pi_1(M)$ is infinite, then h has infinite order [17, Lemma II.4.2], and hence, there is an exact sequence:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1(M)/N \longrightarrow 1.$$

By looking at the presentation above, we see that if the base is oriented, N is central.

When the base is non-oriented, let C be the subgroup of $\pi_1(M)$, of all elements ω written as words on the canonical generators with an even number of occurrences of generators a_1, a_2, \dots, a_g and their inverses. According to the relators of $\pi_1(M)$, obviously, such a fact does not depend on the word chosen in the class of ω . One easily shows that C has index 2 in $\pi_1(M)$, and that C is the centralizer of any non-trivial element of N , while, for all $u \notin C$, $u.h.u^{-1} = h^{-1}$. When the base is oriented, just set $C = \pi_1(M)$; obviously, C is the centralizer of any element of N . This combinatorial definition of the subgroup C , agrees with the topological definition of the *canonical subgroup* of $\pi_1(M)$, as seen in [17].

Let us focus—as an example—on the I -twisted bundle over $\mathbb{K}\mathbb{B}_2$ (which we shall call K) in order to recall elementary facts needed later. The group of K is the group of the Klein bottle $\pi_1(K) = \langle a, b \mid a.b.a^{-1} = b^{-1} \rangle$. Its boundary consists of one toroidal component, and the boundary subgroup is the (free abelian of rank 2) subgroup of index 2 of $\pi_1(K)$: $\pi_1(\partial K) = \langle a^2, b \rangle$.

One can endow K with two Seifert fibrations. The first has a Möbius band as a base, and no exceptional fiber. The class of a regular fiber is b . In this case N is not central, and the canonical subgroup is $\langle a^2, b \rangle$. The second Seifert fibration has a disk as base, and two exceptional fibers of index 2. The class of a regular fiber is a^2 , N is central, and the canonical subgroup is the whole group $\pi_1(K)$.

4.2. Algebraic properties in the pieces

To proceed we first need to establish an algebraic property which is essential to the reduction. Recall that if M is a manifold with a non-empty boundary, and \mathcal{T} is a connected component of ∂M , the canonical embedding $i : \mathcal{T} \hookrightarrow M$ defines a conjugacy class of subgroups of $\pi_1(M)$: the subgroup $T = i_*(\pi_1(\mathcal{T}))$ depends on the choice of a path from the base point of M to the base point of \mathcal{T} . Choosing another such path changes T in gTg^{-1} for some $g \in \pi_1(M)$. Each element of the conjugacy class of T will be called a *boundary subgroup* of $\pi_1(M)$ associated to \mathcal{T} .

Note that one easily verifies that if M is Haken and not homeomorphic to a thickened surface, then boundary subgroups associated to distinct boundary components are non-conjugate.

Proposition 4.1. *Suppose M is a piece obtained in a non-trivial JSJ decomposition of a Haken closed manifold which is not an $S^1 \times S^1$ -bundle over S^1 . Fix for each component of ∂M a boundary subgroup T_i .*

- *If M is hyperbolic, and T_1, T_2 are two non-conjugate boundary subgroups of $\pi_1(M)$, then no non-trivial element of T_1 is conjugate in $\pi_1(M)$ to an element of T_2 . For any boundary subgroup T_1 , if two elements $t, t' \in T_1$ are conjugate by an element $u \in \pi_1(M)$, then necessarily $t = t'$ and $u \in T_1$.*
- *If M is a Seifert fibered space, and is not the twisted I -bundle over the Klein bottle, it admits a unique fibration. We call h the class in $\pi_1(M)$ of a regular fiber and C the canonical subgroup. If T_1, T_2 are*

two non-conjugate boundary subgroups, then $\langle h \rangle \subset T_1 \cap T_2$, and if $v \in \pi_1(M)$ conjugates $t_1 \in T_1$ into $t_2 \in T_2$, then $t_1, t_2 \in \langle h \rangle$ and $t_1 = t_2^{\pm 1}$, with $t_1 = t_2$ precisely when $v \in C$. For any boundary subgroup T_1 , if $t, t' \in T_1$ are conjugate by an element $u \in \pi_1(M)$, then either $t, t' \in \langle h \rangle$ and $t' = t^{\pm 1}$ with $t = t'$ exactly when $u \in C$, or $t = t'$ and $u \in T_1$.

- If M is the twisted I -bundle over the Klein bottle, then $\pi_1(M) = \langle a, b \mid aba^{-1} = b^{-1} \rangle$, and $\pi_1(\partial M) = \langle a^2, b \rangle$. Two elements of $\pi_1(\partial M)$, $a^{2n}b^p$ and $a^{2m}b^q$, are conjugate in $\pi_1(M)$, if and only if $n = m$ and $p = \pm q$.

Proof. Note first that M is a boundary-irreducible Haken manifold, with a non-empty boundary consisting of tori. So M cannot be homeomorphic to $S^1 \times D^2$. Moreover, according to Lemma 2.2, M is not homeomorphic to $S^1 \times S^1 \times I$.

The third case is easy to verify from the presentation given. We leave this as an exercise for the reader. We will prove the two remaining cases separately.

Suppose first that M is hyperbolic. Its fundamental group $\pi_1(M)$ is a torsion-free discrete subgroup of $PSL(2, \mathbb{C})$, which acts by isometries on \mathbb{H}^3 . This action naturally extends to $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. Each boundary subgroup T_i corresponds to a maximal parabolic subgroup of $\pi_1(M)$, with the limit point being the cusp point $p_i \in \partial\mathbb{H}^3$ (i.e. each element of T_i is parabolic and fixes p_i and conversely each parabolic element which fixes p_i is in T_i ; cf. [30, Section 12.2]). Suppose that u in $\pi_1(M)$ conjugates two elements of T_1 ; then u must fix $p_i \in \partial\mathbb{H}^3$. According to Theorem 5.5.4 of [30], u cannot be loxodromic, and hence is parabolic, so $u \in T_1$, which proves the second part of the assertion.

Now suppose T_1, T_2 are two distinct boundary subgroups, characterized by two (distinct) cusp points p_1 and p_2 . If an element u conjugates two non-trivial elements of T_1 and T_2 , then $u.p_1 = p_2$. So, the fact that this cannot occur is a direct implication of the well-known fact that connected components of ∂M are in 1–1 correspondence with orbits under the action of $\pi_1(M)$ of the set of cusp points (cf. [30, Section 12.2]), which concludes the proof in this case.

Suppose now that M is a Seifert fibered space. By hypothesis, M has a non-empty boundary, and is neither $D^2 \times I$ nor the twisted I -bundle over $\mathbb{K}\mathbb{B}_2$, and hence admits a unique fibration. Then $\langle h \rangle$ is an infinite cyclic normal subgroup of $\pi_1(M)$, which does not depend on the regular fiber considered (cf. [17, Lemma II.4.2]). Moreover, any component \mathcal{F} of $\pi_1(M)$ is trivially fibered by regular fibers. Hence any (free abelian of rank two) boundary subgroup of $\pi_1(M)$ contains $\langle h \rangle$ as a subgroup, which proves the beginning of the assertion.

Now suppose that two non-trivial elements t_1, t_2 in the respective boundary subgroups T_1, T_2 (possibly $T_1 = T_2$) are conjugate in $\pi_1(M)$. This gives rise to a map of pairs $f : (S^1 \times I, \partial(S^1 \times I)) \rightarrow (M, \partial M)$ such that its restrictions on $S^1 \times 0$ and $S^1 \times 1$ are non-contractible loops τ_1, τ_2 representing, respectively, t_1 and t_2 .

If this map is essential, according to Lemma II.2.8 of [17], τ_1, τ_2 are homotopic in ∂M to powers of regular fibers, hence $t_1, t_2 \in \langle h \rangle$. Now, it appears clearly from the presentation of $\pi_1(M)$ (cf. Section 4.1), that for all $u \in \pi_1(M)$, $uhu^{-1} = h^\varepsilon$, with $\varepsilon = \pm 1$. Hence, in this case $t_1 = t_2^\varepsilon$ for some $\varepsilon = \pm 1$. Moreover, C is the centralizer of any non-trivial element of $\langle h \rangle$, and so with these notations, $\varepsilon = 1$ precisely when $u \in C$.

If the map is not essential, f is homotopic rel. $\partial(S^1 \times I)$, with a map $g : S^1 \times I \rightarrow \partial M$. Necessarily, τ_1 and τ_2 are in the same component of ∂M , and homotopic in this component. Hence, $T_1 = T_2$ and t_1, t_2 are conjugate in T_1 , but since T_1 is abelian, $t_1 = t_2$, which concludes the proof of the assertion. \square

4.3. Acylindricity of the JSJ decomposition

In this section, M stands for a haken closed manifold which is neither an $S^1 \times S^1$ -bundle over S^1 , nor obtained by gluing two twisted I -bundles over \mathbb{KB}_2 .

As a first consequence of the lack of annuli stated in the previous paragraph, we establish an essential result which asserts the acylindricity (in the sense of Sela [35]) of the Bass–Serre tree associated to the JSJ decomposition. Roughly speaking, it means that there exists $K > 0$, such that if two curves lying in two pieces are freely homotopic in M , then any homotopy between them can be deformed to make contact the JSJ surface at most K times.

Suppose (\mathcal{G}, X) is the JSJ splitting of $\pi_1(M)$, and $u, v \in \pi_1(M)$ are conjugate elements lying in vertex groups. According to Theorem 3.1 (i), (ii), there exists a path $(v_{\alpha_0}, e_{\beta_1}, \dots, e_{\beta_p}, v_{\alpha_p})$ ($p \geq 0$), and a sequence (c_1, c_2, \dots, c_p) following the conclusion of the theorem. We will say this sequence is *reduced* if whenever $e_{\beta_i} = \bar{e}_{\beta_{i+1}}$, then c_i^+ and c_{i+1}^- are non-conjugate in $G(e_{\beta_i})^+$ (otherwise the sequence can be shortened). The integer p is called the *length* of the sequence.

We will say that the JSJ splitting is *k-acylindrical*, if whenever $u, v \in \pi_1(M)$ are conjugate elements lying in vertex groups, any reduced sequence given by the Theorem 3.1 has length at most k . It is an easy exercise to verify that one recovers the original definition of Sela [35].

Lemma 4.1. *Let M be as above. The JSJ splitting of $\pi_1(M)$ is 4-acylindrical.*

Proof. Let (\mathcal{G}, X) denote the JSJ splitting of $\pi_1(M)$. If the splitting is trivial (i.e. X is reduced to a point), then obviously it is 0-acylindrical and the conclusion follows, so that we will further suppose that this case does not occur. According to Theorem 3.1, there exists a path in X : $\mathcal{P} = (v_{\alpha_0}, e_{\beta_1}, v_{\alpha_1}, \dots, e_{\beta_p}, v_{\alpha_p})$, and a sequence (c_1, c_2, \dots, c_p) with $\forall i = 1, 2, \dots, p$, $c_i \in G(e_{\beta_i})$, $u \in G(v_{\alpha_0})$, $v \in G(v_{\alpha_p})$, such that $u \sim c_1^- = \varphi_{\beta_1}^-(c_1)$ in $G(v_{\alpha_0})$, $v \sim c_p^+$ in $G(v_{\alpha_p})$, and for $i = 1, 2, \dots, p-1$, $c_i^+ \sim c_{i+1}^-$ in $G(v_{\alpha_i})$. We can also suppose, if for some i , $e_{\beta_i} = \bar{e}_{\beta_{i+1}}$, that c_i^+ and c_{i+1}^- are not conjugate in $G(e_{\beta_i})^+ = G(e_{\beta_{i+1}})^-$ because otherwise one can shorten the path and the sequence, while continuing to verify the above conditions. Hence we can apply Proposition 4.1, which implies that for $i = 1, 2, \dots, p-1$, none of the $M(v_{\alpha_i})$ is hyperbolic, and hence they are Seifert fiber spaces. Note also that $M(v_{\alpha_i})$ and $M(v_{\alpha_{i+1}})$ cannot both be twisted I -bundles over \mathbb{KB}_2 , because otherwise, M would be obtained by gluing two twisted I -bundles over \mathbb{KB}_2 along their boundary. Moreover, for $i = 1, 2, \dots, p-2$, $M(v_{\alpha_i})$ and $M(v_{\alpha_{i+1}})$ cannot both be non- I -twisted bundles over \mathbb{KB}_2 : otherwise, $c_i^+ \sim c_{i+1}^-$ in $G(v_{\alpha_i})$ and $c_{i+1}^+ \sim c_{i+2}^-$ in $G(v_{\alpha_{i+1}})$; hence according to Proposition 4.1, c_{i+1}^- and c_{i+1}^+ must lie, respectively, in the fibers of $G(v_{\alpha_i})$ and $G(v_{\alpha_{i+1}})$, but this fact contradicts Lemma 2.1. Hence, for $i = 1, 2, \dots, p-1$, the successive pieces $M(v_{\alpha_i})$ must be alternatively I -twisted bundles over \mathbb{KB}_2 , and Seifert pieces which are not I -twisted bundles over \mathbb{KB}_2 . In fact, since twisted I -bundles have only one boundary component, if for some $i = 1, 2, \dots, p-1$, $M(v_{\alpha_i})$ is a twisted I -bundle over \mathbb{KB}_2 , then necessarily $e_{\beta_i} = \bar{e}_{\beta_{i+1}}$, and $v_{\alpha_{i-1}} = v_{\alpha_{i+1}}$; hence the pieces $M(v_{\alpha_1}), M(v_{\alpha_2}), \dots, M(v_{\alpha_{p-1}})$ are alternatively the same Seifert piece which is not a twisted I -bundle over \mathbb{KB}_2 , and possibly several twisted I -bundles over \mathbb{KB}_2 each of them being glued to this last non-twisted I -bundle piece.

Now suppose $p \geq 5$; then without loss of generality, one can suppose that $v_{\alpha_1} = v_{\alpha_3}$, $M(v_{\alpha_1}) = M(v_{\alpha_3})$ is a Seifert piece which is not a twisted I -bundle over \mathbb{KB}_2 , while $M(v_{\alpha_2})$ is a twisted I -bundle over \mathbb{KB}_2 . Now, $c_1^+ \sim c_2^-$ in $G(v_{\alpha_1})$, $c_2^+ \sim c_3^-$ in $G(v_{\alpha_2})$, and $c_3^+ \sim c_4^-$ in $G(v_{\alpha_1})$. Consider the canonical presentation

$\langle a, b | aba^{-1} = b^{-1} \rangle$ of $G(v_{x_2})$; then, according to Lemma 4.1, for some integers n, m , $c_2^+ = a^n b^m$ and $c_3^- = a^n b^{-m}$ with $m \neq 0$ (otherwise they would be conjugate in $G(e_{\beta_2})^+ = \langle a^2, b \rangle$). But this last lemma also implies that $c_2^- = \varphi_{\beta_2}^{-1}(c_2^+)$ and $c_3^+ = \varphi_{\beta_3}(c_3^-)$ both lie in the fiber of $G(v_{x_1})$. This is possible only if φ_{β_2} sends the fiber to the subgroup $\langle b \rangle$ of $G(v_{x_2})$. But this would contradict Lemma 2.1, since $\langle b \rangle$ is the fiber of $G(v_{a_2})$ for one of its two Seifert fibrations. Hence $p \leq 4$, which concludes the proof. \square

4.4. Processing cyclically reduced forms

Suppose M is a Haken closed manifold whose group $\pi_1(M)$ is given by its canonical presentation. We have described in Section 3.1, how one can, given a word ω on the canonical generators of $\pi_1(M)$, $\omega \neq 1$, find a cyclically reduced form whose label is a conjugate of ω . This process is constructive. In order to make use of Theorem 3.1 in a constructive way, we need to have an algorithmic process to transform an arbitrary form into a cyclically reduced one. This is the aim of this section.

We claim that in order to perform such a process, it suffices to have a solution to the generalized word problem of T in $\pi_1(N)$, for any piece N and any boundary subgroup T of $\pi_1(N)$, that is, an algorithm which decides for any $u \in \pi_1(N)$ given as a word on the generators of $\pi_1(N)$, whether $u \in T$ or not. For, suppose (\mathcal{C}, μ) is a form with label ω . Any time \mathcal{C} contains a sub-path of the form (v', e, v, \bar{e}, v') , check with a solution to the generalized word problem of $G(e)^+$ in $\pi_1(M(v))$ whether the label u of the vertex v is an element of $G(e)^+$. Then in this case, replace in \mathcal{C} the subpath (v', e, v, \bar{e}, v') with label (u_1, u, u_2) with (v') and label $u_1 \cdot \varphi_e^{-1}(u) \cdot u_2$ which is an element of $\pi_1(M(v'))$ (if \mathcal{C} has length 2, replace (v', e, v, \bar{e}) with (v') and label $u_1 \cdot \varphi_e^{-1}(u)$). One obtains a shorter cyclic form, with labelling a word equal in $\pi_1(M)$ to ω . Perform this process as long as it is possible with (\mathcal{C}, μ) and all of its cyclic conjugates. Since the length strictly decreases it will necessarily stop. The cyclic form obtained is cyclically reduced. Its label is a conjugate in $\pi_1(M)$ of the label ω of (\mathcal{C}, μ) .

A solution to the generalized word problem can be easily found using the solution to the word problem in the π_1 of the pieces, as well as the algebraic properties of boundary subgroups, seen in the last section.

Proposition 4.2. *Suppose N is a piece obtained in the non-trivial decomposition of a Haken closed manifold. Suppose T is any boundary subgroup of $\pi_1(N)$. Then one can effectively decide for any $u \in \pi_1(N)$, whether $u \in T$ or not; in other words, the generalized word problem of T in $\pi_1(M)$ is solvable.*

Proof. Obviously, we can suppose that N is not homeomorphic to $S^1 \times S^1 \times I$. Suppose N is hyperbolic. Then according to Proposition 4.1, for any non-trivial element t of T , the centralizer of t in $\pi_1(N)$ is precisely T . Suppose N is a Seifert fibered space, and is not the twisted I -bundle over $\mathbb{K}\mathbb{B}_2$. Then, with Proposition 4.1, for any element $t \in T$ which does not lie in the fiber $\langle h \rangle$, the centralizer of t in $\pi_1(N)$ is precisely T . In each case, to decide whether an element $u \in \pi_1(N)$ lies in T , it suffices to apply the solution to the word problem in $\pi_1(M)$ (cf. [40]) to decide whether $ut = tu$ or not for such a $t \in T$.

In the case of the twisted I -bundle over $\mathbb{K}\mathbb{B}_2$, an element in $\pi_1(N) = \langle a, b | aba^{-1} = b^{-1} \rangle$ lies in $\pi_1(\partial N)$ exactly when it is written as a word with an even number of occurrence of the generator a or its inverse, which can be easily verified. \square

As explained above, one immediately obtains the corollary:

Corollary 4.1. *Let M be a Haken closed manifold. If $\pi_1(M)$ is given by its canonical presentation, then one can, given a word ω on the generators, algorithmically find a cyclically reduced form for ω .*

4.5. The core of the algorithm

We can now give the main algorithm to solve the conjugacy problem in the group $\pi_1(M)$ of a Haken closed manifold M . This algorithm uses solutions to the conjugacy, the boundary parallelism, and the 2-cosets problems in the groups of each piece, as well as a solution to the word problem in $\pi_1(M)$. Hence the conjugacy problem in $\pi_1(M)$ reduces to the conjugacy, boundary parallelism, and 2-cosets problems in the groups of the pieces, which will be solved later. We first need to establish two lemmas, to define correctly the boundary parallelism and 2-cosets problems, and also to handle them. The following two lemmas are essential, and are direct consequences of the algebraic property established in Proposition 4.1.

In the following N stands for a piece in the JSJ decomposition of a Haken closed manifold M .

Lemma 4.2. *For any boundary subgroup T of $\pi_1(N)$, and for any element $\omega \in \pi_1(N)$, define the subset of T , $C_T(\omega) = \{c \in T \mid \omega \sim c \text{ in } \pi_1(N)\}$.*

- If N is hyperbolic, then $C_T(\omega)$ is either empty or a singleton.
- If N admits a Seifert fibration then $C_T(\omega)$ has cardinality at most 2.

Proof. Suppose there exists two distinct elements c_1 and c_2 in $C_T(\omega)$. Then c_1 and c_2 are conjugate in $\pi_1(N)$. According to Proposition 4.1, this cannot happen if N is hyperbolic, which proves the first assertion. So N must admit a Seifert fibration. Suppose first that N is not the twisted I -bundle over \mathbb{KB}_2 . Then necessarily (Proposition 4.1) $c_1 = c_2^{\pm 1}$, and hence $C_T(\omega)$ is of cardinality at most 2. In the case of the twisted I -bundle over \mathbb{KB}_2 , $c_1 = a^{2n}b^p$ and $c_2 = a^{2n}b^{\pm q}$ for some integers n, p , and the same conclusion holds. \square

Lemma 4.3. *For any boundary subgroup T, T' of $\pi_1(N)$ (possibly $T = T'$), and for any element $\omega, \omega' \in \pi_1(N)$, define the subset of $T \times T'$, $C_{T,T'}(\omega, \omega') = \{(c, c') \in T \times T' \mid \omega = c.\omega'.c'\}$. Suppose moreover that in case $T = T'$, ω' does not lie in T .*

- If N is hyperbolic, $C_{T,T'}(\omega, \omega')$ is either empty or a singleton.
- If N admits a Seifert fibration, and is not the twisted I -bundle over \mathbb{KB}_2 , let $\langle h \rangle$ denote the fiber and C the canonical subgroup of $\pi_1(N)$. Then either $C_{T,T'}(\omega, \omega')$ is empty or $C_{T,T'}(\omega, \omega') = \{(ch^n, c'h^{-\varepsilon.n}) \mid n \in \mathbb{Z}\}$ with $\varepsilon = \pm 1$ according to whether $\omega' \in C$ or not.
- If N is the twisted I -bundle over \mathbb{KB}_2 , then either $\omega \in T$ and $C_{T,T}(\omega, \omega')$ is empty, or $C_{T,T'}(\omega, \omega') = \{(\omega\omega'^{-1}.a^{2n}b^p, a^{-2n}b^p) \mid n, p \in \mathbb{Z}\}$.

Remark. If $T = T'$ and $\omega' \in T$, then either $\omega \in T$ and $C_{T,T}(\omega, \omega') = \{(t.\omega\omega'^{-1}, t^{-1}) \mid t \in T\}$ or $C_{T,T}(\omega, \omega')$ is empty. This is an obvious consequence of the fact that T is abelian.

Proof of Lemma 4.3. Suppose there exist two distinct elements (c_1, c'_1) and (c_2, c'_2) in $C_{T,T'}(\omega, \omega')$. Then $c_2^{-1}c_1 \in T$ is conjugate in $\pi_1(N)$ to $c'_2c'_1{}^{-1} \in T'$ by ω'^{-1} . If N is hyperbolic, according to Proposition 4.1, necessarily $c_2^{-1}c_1 = c'_2c'_1{}^{-1} = 1$, which contradicts the fact that (c_1, c'_1) and (c_2, c'_2) are distinct. Thus in the hyperbolic case, $C_{T,T'}(\omega, \omega')$ has cardinality at most 1.

If N admits a Seifert fibration and is not the twisted I -bundle over \mathbb{KB}_2 , then necessarily (Proposition 4.1) for some integer n , $c_2^{-1}c_1 = h^n$ and $c'_2c'_1{}^{-1} = h^{\varepsilon.n}$, with $\varepsilon = 1$ or -1 , according to whether $\omega' \in C$

or not. Hence, $c_2 = c_1 \cdot h^{-n}$ and $c'_2 = h^{\varepsilon \cdot n} c'_1 = c'_1 h^{\varepsilon \cdot n}$. Reciprocally, if (c_1, c'_1) lies in $C_{T, T'}(\omega, \omega')$, then so does any couple of the form $(c_1 h^n, c'_1 h^{-\varepsilon \cdot n})$. Hence, if $C_{T, T'}(\omega, \omega')$ is non-empty, it must be of the form $\{(c h^n, c' h^{-\varepsilon \cdot n}) \mid n \in \mathbb{Z}\}$ for some $(c, c') \in T \times T'$, with $\varepsilon = 1$ or -1 , according to whether $\omega' \in C$ or not.

If N is the twisted I -bundle over \mathbb{KB}_2 , then $\pi_1(N) = \langle a, b \mid aba^{-1} = b^{-1} \rangle$, and $T = \langle a^2, b \rangle$ is an abelian (normal) subgroup of index 2. If $w = c\omega'c'$ for some $t, t' \in T$, then if ω lies in T , so does ω' . Hence if $\omega \in T$, such an equality cannot occur. Since ω and ω' both lie outside T which has an index 2, $\omega\omega'^{-1} \in T$, and thus obviously $(\omega\omega'^{-1}, 1) \in C_{T, T}(\omega, \omega')$.

An element of $\pi_1(N)$ lies in T , exactly when it can be written in the form $a^{2n}b^q$ for some integers n, q . Since $\omega' \notin T$, one easily verify that for all integers n, p , the equation $\omega'a^{2n}b^p = a^{2n}b^{-p}\omega'$ holds. Proceed as above, and suppose that $C_{T, T}(\omega, \omega')$ contains two distinct elements (c_1, c'_1) and (c_2, c'_2) . Necessarily, ω conjugates $c'_2c'_1{}^{-1}$ into $c_2^{-1}c_1$, and hence with Proposition 4.1, $c'_2c'_1{}^{-1} = a^{2n}b^p$ and $c_2^{-1}c_1 = a^{2n}b^{-p}$ for some integers n, p . So, $c'_2 = a^{2n}b^p c'_1 = c'_1 a^{2n}b^p$, and $c_2 = c_1 a^{-2n}b^p$. Then, the elements of $C_{T, T'}(\omega, \omega')$ are all those of the form $(\omega\omega'^{-1} \cdot a^{2n}b^p, a^{-2n}b^p)$ for some integers n, p , which concludes the proof. \square

Now we consider in the group $\pi_1(N)$ of any piece N , two decision problems: the boundary parallelism problem, and the 2-coset problem.

The boundary parallelism problem: Let T be a boundary subgroup of $\pi_1(N)$. Construct an algorithm which for any $\omega \in \pi_1(N)$, determines $C_T(\omega)$, i.e. find all elements of T conjugate to ω in $\pi_1(N)$.

The 2-coset problem: Let T, T' be two boundary subgroups of $\pi_1(N)$ (possibly $T = T'$). Construct an algorithm which for any couple of elements $\omega, \omega' \in \pi_1(N)$, determines $C_{T, T'}(\omega, \omega')$, i.e. finds all the couples $(c, c') \in T \times T'$ such that $\omega = c \cdot \omega' \cdot c'$ in $\pi_1(N)$.

We are now able to show that if one can solve the conjugacy, boundary parallelism and 2-cosets problems in the groups of all the pieces obtained in the JSJ decomposition of M , then one can solve the conjugacy problem in $\pi_1(M)$. The cases of $S^1 \times S^1$ -bundles over S^1 , and of two twisted I -bundles over \mathbb{KB}_2 glued along their boundary, are rather easy to deal with, and a solution to the conjugacy problem in their respective groups will be sketched in Section 7.

Recall that we suppose that a canonical presentation of $\pi_1(M)$ is given. Elements of $\pi_1(M)$ are given as words on the canonical generators.

Theorem 4.1. *The conjugacy problem in the group of a Haken closed manifold M which is neither an $S^1 \times S^1$ -bundle over S^1 nor obtained by gluing two twisted I -bundles over \mathbb{KB}_2 along their boundary reduces to conjugacy problems, boundary parallelism problems, and 2-cosets problems, in the groups of the pieces obtained. In other words, if one can solve these three problems in each of the groups of the pieces, then one can solve the conjugacy problem in $\pi_1(M)$.*

Proof. We will suppose that each piece admits a solution to the last three problems, and solve the conjugacy problem in $\pi_1(M)$. Suppose we are given two words ω and ω' on the canonical generators and want to decide whether or not ω and ω' are conjugate in $\pi_1(M)$.

First, we use Corollary 4.1 to find cyclically reduced forms (\mathcal{C}, μ) and (\mathcal{C}', μ') , respectively, associated with ω and ω' . Without loss of generality we will suppose that their labels are precisely ω and ω' . According to Theorem 3.1 we can also suppose that the cyclically reduced forms obtained both have the same length, because otherwise ω, ω' are definitely not conjugate in $\pi_1(M)$.

Suppose first that (\mathcal{C}, μ) and (\mathcal{C}', μ') both have length 0. This happens when the paths \mathcal{C} and \mathcal{C}' of X are reduced to points, say $\mathcal{C} = (v)$, $\mathcal{C}' = (v')$, and thus ω and ω' lie in the respective vertex groups $G(v)$ and $G(v')$.

If $v = v'$, apply the solution to the conjugacy problem in $G(v)$ to decide whether or not ω and ω' are conjugate in $G(v)$. In the former case, ω and ω' are conjugate in $\pi_1(M)$, but in the latter case one cannot conclude anything yet, and one needs to apply the general process as described below.

For any boundary subgroup T of $G(v)$, use the solution in $G(v)$ to the boundary parallelism problem to find all elements in T conjugate to ω . According to Lemma 4.2, one finds at most two such elements. Apply the same process with ω' in $G(v')$. One eventually finds $c \in G(e)^- \subset G(v)$ and $c' \in G(e')^- \subset G(v')$, two respective conjugates of ω and ω' . Then apply the same process with $\varphi_e(c) \in G(e)^+ \subset G(\mathbf{e}(e))$ and $\varphi_{e'}(c') \in G(e')^+ \subset G(\mathbf{e}(e'))$, and successively with all the boundary conjugates obtained, to eventually obtain a labelled path from v to v' as in Theorem 3.1(ii), in which case ω and ω' are conjugate in $\pi_1(M)$. Since at each step one finds at most two boundary conjugates, and since according to Lemma 4.1 if such a path exists there exists one with length at most 4, the process must terminate. According to Theorem 3.1, if ω and ω' are not conjugate in some vertex group $G(v)$, and if one cannot find such a path, then ω and ω' are definitely not conjugate in $\pi_1(M)$.

Suppose now that (\mathcal{C}, μ) and (\mathcal{C}', μ') both have a length greater than 0. Up to cyclic conjugation of (\mathcal{C}', μ') we can suppose that $\mathcal{C} = \mathcal{C}' = (v_{\sigma_1}, e_{\tau_1}, \dots, v_{\sigma_n}, e_{\tau_n})$, because otherwise, according to Theorem 3.1(iii), ω and ω' are definitely not conjugate in $\pi_1(M)$.

First suppose that the path \mathcal{C} passes through a vertex v whose corresponding piece $M(v)$ is hyperbolic. By possibly considering cyclic conjugates of \mathcal{C} , one can suppose that this arises for $M(v_{\sigma_1})$. According to Theorem 3.1(iii), if ω and ω' are conjugate in $\pi_1(M)$, then necessarily, there exists $c_n^+ \in G(e_{\tau_n})^+$ which conjugates ω' into ω , and moreover there also exists $c_1^- \in G(e_{\tau_1})^-$ such that $\mu_1 = c_n^+ \cdot \mu'_1 \cdot (c_1^-)^{-1}$ in $G(v_1) = \pi_1(M(v_1))$. Then, since μ_1 and μ'_1 are given, using the solution to the 2-cosets problem in $\pi_1(M(v_1))$ one finds at most one couple of solutions (c_n^+, c_1^-) (cf. Lemma 4.3). Once we know $c_n^+ \in G(e_{\tau_n})^+$, we can use a solution to the word problem in $\pi_1(M)$ (cf. [40]), to decide whether or not $\omega = c_n^+ \cdot \omega' \cdot (c_n^+)^{-1}$ in $\pi_1(M)$. In the former case, obviously $\omega \sim \omega'$, but in the latter case, before concluding one needs to apply the same process with all possible cyclic conjugates (\mathcal{C}'', μ'') of (\mathcal{C}', μ') such that $\mathcal{C}'' = \mathcal{C}'$. Since they are of finite number, according to Theorem 3.1, one can finally decide whether $\omega \sim \omega'$ or not.

Now suppose that the path \mathcal{C} only passes through vertices whose corresponding pieces are Seifert fibered spaces. Suppose first that \mathcal{C} is of length more than 1 and contains a subpath of length 1 (v_1, e, v_2) where neither $M(v_1)$ nor $M(v_2)$ is homeomorphic to the twisted I -bundle over $\mathbb{K}\mathbb{B}_2$. Up to cyclic conjugations we can suppose that this condition arises for the initial subpath $(v_{\sigma_1}, t_{\tau_1}, v_{\sigma_2})$ of \mathcal{C} . According to Theorem 3.1, if ω and ω' are conjugate in $\pi_1(M)$, then there exist $c_n^+ \in G(e_{\tau_n})^+$, $c_1^- \in G(e_{\tau_1})^-$ and $c_2^- \in G(e_{\tau_2})^-$, such that

$$\begin{aligned} \omega &= c_n^+ \cdot \omega' \cdot (c_n^+)^{-1} \quad \text{in } \pi_1(M), \\ \mu_1 &= c_n^+ \cdot \mu'_1 \cdot (c_1^-)^{-1} \quad \text{in } \pi_1(M(v_{\sigma_1})), \end{aligned} \tag{1}$$

$$\mu_2 = c_1^+ \cdot \mu'_2 \cdot (c_2^-)^{-1} \quad \text{in } \pi_1(M(v_{\sigma_2})). \tag{2}$$

We consider (1) and (2) as equations with respective unknowns the couples (c_n^+, c_1^-) and (c_1^+, c_2^-) . We denote S_1 and S_2 as the sets of couples of solutions. Those sets are either empty or infinite

(cf. Lemma 4.3). In the former case ω and ω' are not conjugate. In the latter case we denote C_1^- and C_1^+ as the subsets of $G(e_{\tau_1})$ defined as the respective images of S_1 and S_2 under the maps $(\varphi_{\tau_1}^-)^{-1} \circ \pi_2$ and $(\varphi_{\tau_1}^+)^{-1} \circ \pi_1$, where π_1, π_2 stand for the canonical first and second projections, and $\varphi_{\tau_1}^- : G(e_{\tau_1}) \rightarrow G(e_{\tau_1})^-, \varphi_{\tau_1}^+ : G(e_{\tau_1}) \rightarrow G(e_{\tau_1})^+$ are the monomorphisms associated with the edge e_{τ_1} . According to Lemma 4.3, C_1^- is a 1-dimensional affine subset of the \mathbb{Z} -module $G(e_{\tau_1}) \simeq \mathbb{Z} \oplus \mathbb{Z}$, with slope $(\varphi_{\tau_1}^-)^{-1}(h_1)$ where h_1 is the class of a regular fiber in $M(v_{\sigma_1})$, and similarly C_1^+ is a 1-dimensional affine subset of $G(e_{\tau_1})$, with slope $(\varphi_{\tau_1}^+)^{-1}(h_2)$ where h_2 is the class of a regular fiber in $M(v_{\sigma_2})$. The possible $c_1 = (\varphi_{\tau_1}^-)^{-1}(c_1^-) = (\varphi_{\tau_1}^+)^{-1}(c_1^+)$ must lie in $C_1^- \cap C_1^+$, and hence, are solutions in $\mathbb{Z} \oplus \mathbb{Z}$ of a system (S) of two affine equations. The key point is that, according to Lemma 2.1, C_1^- and C_1^+ must have distinct slopes, and so the system (S) admits at most one solution, which one can easily determine. This gives at most one element c_1^- , which, according to Lemma 4.3, allows the determination of at most one potential element $c_n^+ \in G(e_{\tau_n})^+$ which may conjugate ω' in ω in $\pi_1(M)$. Now using a solution to the word problem in $\pi_1(M)$, we only need to verify whether $\omega = c_n^+ \cdot \omega' \cdot (c_n^+)^{-1}$ in $\pi_1(M)$. If this does not happen, then apply the same process to all the cyclic conjugates of (\mathcal{C}', μ') , whose underlying loops are equal to \mathcal{C} (they are of finite number). If one does not find in such a way an element $c_n^+ \in G(e_{\tau_n})^+$ which conjugates ω' in ω , then, according to Theorem 3.1(iii), ω and ω' are not conjugate in $\pi_1(M)$.

Suppose now that \mathcal{C} and \mathcal{C}' have length one. Then $\mathcal{C} = \mathcal{C}' = (v, e)$ and so the edge e both starts and ends in v . Hence the piece $M(v)$ has at least two boundary components, and then cannot be homeomorphic to the twisted I -bundle over \mathbb{KB}_2 . Now, according to Theorem 3.1, $\omega = \mu_1 \cdot t_e$ and $\omega' = \mu'_1 \cdot t_e$, and $\omega \sim \omega'$ if and only if there exists $c^+ \in G(e)^+$ such that

$$\begin{aligned} \mu_1 \cdot t_e &= c^+ \cdot \mu'_1 \cdot t_e \cdot (c^+)^{-1} \\ &= c^+ \cdot \mu'_1 \cdot \varphi_e^{-1}((c^+)^{-1}) \cdot t_e \\ \Leftrightarrow \mu_1 &= c^+ \cdot \mu'_1 \cdot \varphi_e^{-1}((c^+)^{-1}) \quad \text{in } G(v). \end{aligned}$$

Use the solution to the 2-coset problem in $G(v)$ to find all couples (c, c_1) with $c \in G(e)^+, c_1 \in G(e)^-$, such that $\mu_1 = c \cdot \mu'_1 \cdot c_1$. According to Lemma 4.3, the set of solutions S is either empty or $S = \{(d \cdot h^n, d_1 \cdot h^{-\varepsilon \cdot n}) | n \in \mathbb{Z}\}$, where h is the class of a regular fiber, and $\varepsilon = \pm 1$ according to whether μ'_1 (and μ_1) is in the canonical subgroup C of $\pi_1(M(v))$ or not. If $S = \emptyset$, definitely ω and ω' are not conjugate. Otherwise, $\omega \sim \omega'$ if and only if there exists $(c, c_1) \in S$ such that $\varphi_e(c_1) = c^{-1}$, if and only if there exists $n \in \mathbb{Z}$, such that in $G(e)^+$,

$$\begin{aligned} \varphi_e(d_1 \cdot h^{-\varepsilon \cdot n}) &= h^{-n} \cdot d^{-1} \\ \Leftrightarrow \varphi_e(d_1) \cdot d &= h^{-n} \cdot \varphi_e(h^{\varepsilon \cdot n}) \\ \Leftrightarrow \varphi_e(d_1) \cdot d &= (h^{-1} \cdot \varphi_e(h^{\varepsilon}))^n. \end{aligned}$$

Now $G(e)^+ \simeq \mathbb{Z} \oplus \mathbb{Z}$, and once a base is given, one can write in additive notations, $\varphi_e(d_1) \cdot d = (a, b)$ and $h^{-1} \cdot \varphi_e(h^{\varepsilon}) = (\alpha, \beta)$, which are given and do not depend on n . The relation becomes $(a, b) = n \cdot (\alpha, \beta)$, and hence $\omega \sim \omega'$ if and only if the two vectors (a, b) and (α, β) of $\mathbb{Z} \oplus \mathbb{Z}$ are collinear, which can be checked easily. Thus, one can decide in this case whether $\omega \sim \omega'$ or not.

Now, the only remaining case is when \mathcal{C} is of length greater than one, and such that for any sub-path of length 1 (v, e, v') of \mathcal{C} , either $M(v)$ or $M(v')$ is homeomorphic to the twisted I -bundle over \mathbb{KB}_2 . Note that since the twisted I -bundle over \mathbb{KB}_2 has only one boundary component, if the vertex v appearing in

\mathcal{C} is such that $M(v)$ is homeomorphic to this last manifold, then it must necessarily appear in a sub-path of \mathcal{C} of the form (e, v, \bar{e}) . Moreover, \mathcal{C} cannot contain a sub-path of the form (v_0, e, v_1, \bar{e}) where both $M(v_0)$ and $M(v_1)$ are twisted I -bundles over \mathbb{KB}_2 , except when M is obtained by gluing two twisted I -bundles over \mathbb{KB}_2 along their boundary, which has been excluded.

Suppose first that \mathcal{C} has length 2. Then up to cyclic conjugations $\mathcal{C} = \mathcal{C}' = (v_0, e, v_1, \bar{e})$, where $M(v_1)$ is homeomorphic to the twisted I -bundle, and $M(v_0)$ is not. Then, $\omega = \mu_0 \cdot t_e \cdot \mu_1 \cdot t_e^{-1}$ and $\omega' = \mu'_0 \cdot t_e \cdot \mu'_1 \cdot t_e^{-1}$, and according to Theorem 3.1(iii), $\omega \sim \omega'$ if and only if there exists $c_0, c_1 \in G(e)$, such that

$$\begin{aligned} \mu_0 &= c_0^- \cdot \mu'_0 \cdot c_1^- && \text{in } G(v_0), \\ \mu_1 &= c_1^+ \cdot \mu'_1 \cdot (c_0^+)^{-1} && \text{in } G(v_1). \end{aligned}$$

Now use the solution to the 2-cosets problem in $G(v_0)$ to find the subset S of $G(e)^- \times G(e)^-$, of all possible (c_0^-, c_1^-) satisfying the above first equation. According to Lemma 4.3 one obtains $S = \{(d_0 \cdot h^n, d_1 \cdot h^{-\varepsilon \cdot n}) | n \in \mathbb{Z}\}$, where h is the class of a regular fiber of $M(v_0)$ and $\varepsilon = \pm 1$ according to whether μ'_0 lies in the canonical subgroup C or not. Pick the base a^2, b of $G(e)^+$, and using additive notations in $G(e)^+$, note $\varphi_e(h) = (p, q)$, then $c_0^+ = \varphi_e(c_0^-) = (\alpha, \beta) + n \cdot (p, q)$ and $c_1^+ = \varphi_e(c_1^-) = (\gamma, \delta) - \varepsilon \cdot n \cdot (p, q)$, for $n \in \mathbb{Z}$, and for some elements (α, β) and (γ, δ) that one directly finds from S, φ_e , and the base. Now, $\omega \sim \omega'$ exactly when $\mu_1 = c_1^+ \cdot \mu'_1 \cdot (c_0^+)^{-1}$ in $G(v_1)$. According to Lemma 4.3, this happens exactly when c_1^+ is the image of c_0^+ by the transformation of $\mathbb{Z} \oplus \mathbb{Z}$ obtained by composing the linear map which sends $a^2 \rightarrow a^2$, and $b \rightarrow b^{-1}$ followed by the translation of vector $\mu_1 \mu'_1^{-1} = (\lambda, \theta)$. Hence $\omega \sim \omega'$ if and only if there exists an integer solution n of the equation $n \cdot (p, -q) + \varepsilon \cdot n \cdot (p, q) = (\gamma - \alpha - \lambda, \delta + \beta - \theta)$, where $\varepsilon, p, q, \alpha, \beta, \gamma, \delta, \lambda, \theta$ are given, which can be easily checked.

Now suppose \mathcal{C} has length greater than 2. Then up to cyclic conjugation it must contain as an initial sub-path $(v_0, e, v_1, e^{-1}, v_0, \dots)$ where $M(v_1)$ is homeomorphic to the twisted I -bundle over \mathbb{KB}_2 , while $M(v_0)$ is not. According to Theorem 3.1, $\omega \sim \omega'$ if and only if there exists $c_n^+ \in G(e_{\tau_n})^+$ which conjugates ω' to ω . Moreover, necessarily, there exist elements $c_1, c_2 \in G(e)$ and $c_3 \in G(e_{\tau_3})$, such that

$$\begin{aligned} \mu_1 &= c_n^+ \cdot \mu'_1 \cdot c_1^- && \text{in } G(v_0), \\ \mu_2 &= c_1^+ \cdot \mu'_2 \cdot c_2^+ && \text{in } G(v_1), \\ \mu_3 &= c_2^- \cdot \mu'_3 \cdot c_3^- && \text{in } G(v_0) \end{aligned}$$

with $c_1^-, c_2^- \in G(e)^-$, $c_3^- \in G(e_{\tau_3})^-$, and $c_1^+, c_2^+ \in G(e)^+$. Note that C_1 is the set of possible $c_1^+ \in G(e)^+$ such that c_1^- verifies the first equation, and C_2 is the set of possible $c_2^+ \in G(e)^+$ such that $c_2^- = \varphi_e^{-1}(c_2^+)$ satisfies the last equation, and use the solutions to the 2-cosets problem, to find them. We will suppose that they are both non-empty because otherwise ω and ω' are not conjugate. Let h be the class of a regular fiber in $\pi_1(M(v_0))$. Pick the base a^2, b of $G(e)^+$, and use additive notations. Then according to Lemma 4.3, C_1 and C_2 are 1-dimensional affine subsets, $C_1 = (\alpha, \beta) + \mathbb{Z} \cdot (p, q)$ and $C_2 = (\gamma, \delta) + \mathbb{Z} \cdot (p, q)$ of $G(e)^+$, where (p, q) stands for the natural image of h under φ_e . But now, necessarily, if $\omega \sim \omega'$ then $\mu_2 = c_1^+ \cdot \mu'_2 \cdot c_2^+$ in $G(v_1)$. According to Lemma 4.3, this happens exactly when c_1^+ is the image of c_2^+ by the transformation of $\mathbb{Z} \oplus \mathbb{Z}$, composed of the linear transformation defined by $a^2 \rightarrow a^{-2}$ and $b \rightarrow b$ followed by the translation of vector $\mu_1 \mu'_1^{-1} = (\lambda, \theta)$. Hence, if $c_1^+ = (\alpha, \beta) + n \cdot (p, q)$ and $c_2^+ = (\gamma, \delta) + m \cdot (p, q)$ this gives rise to the equation with the unknowns $n, m \in \mathbb{Z}$,

$$n \cdot (p, q) + m \cdot (p, -q) = (\lambda - \gamma - \alpha, \delta + \theta - \beta).$$

Now, according to Lemma 2.1, (p, q) cannot be a regular fiber of $M(v_1)$, and hence (recall the two Seifert fibrations of the twisted I -bundle over \mathbb{KB}_2) neither p nor q is null. Hence this gives rise to a system of two affine equations, which admits at most one couple of integer solutions (n, m) . Now, once we know n , we know c_1^- , and consequently we know c_n^+ (according to Lemma 4.3). To decide whether $\omega \sim \omega'$, it suffices to check with the solution to the word problem in $\pi_1(M)$ whether $\omega = c_n^+ \cdot \omega' \cdot (c_n^+)^{-1}$ or not.

Hence, given ω and ω' in $\pi_1(M)$, by applying this process, one can decide whether $\omega \sim \omega'$ or not. Hence the conjugacy problem in $\pi_1(M)$ is solvable, which concludes the argument. \square

The rest of our work will now consist in finding solutions to the boundary parallelism, 2-cosets, and conjugacy problems in the groups of a Seifert fiber space, or a hyperbolic 3-manifold with finite volume, as well as in solving the conjugacy problem in the few remaining cases of $S^1 \times S^1$ -bundles over S^1 or manifolds obtained by gluing two twisted I -bundles over \mathbb{KB}_2 (cf. Section 7).

5. The case of a Seifert fibered space

This section is devoted to obtaining the algorithms needed in the group of a Seifert fiber space. We focus essentially on the boundary parallelism and 2-cosets problem: almost all Seifert fiber spaces have a biautomatic group, and hence a solvable conjugacy problem; the only remaining case—the one of manifolds modelled on NIL —can be treated easily, and will only be sketched in Section 5.3.

5.1. Preliminaries

Recall that if M is a Seifert fiber space, any regular fiber generates a cyclic normal subgroup N called the *fiber*. Moreover, N is infinite exactly when $\pi_1(M)$ is infinite (cf. Section 4.1).

$$1 \longrightarrow N \longrightarrow \pi_1(M) \longrightarrow \pi_1(M)/N \longrightarrow 1.$$

Note also that the property of having a group which contains a normal cyclic subgroup characterizes among all irreducible 3-manifolds with infinite π_1 those admitting a Seifert fibration (known as the “Seifert fiber space conjecture”, this has been recently solved as a result of a collective work, including Casson, Gabai, Jungreis, Mess and Tukia).

The quotient group $\pi_1(M)/N$ is one of a well-known class of groups, called *Fuchsian groups* in the terminology of [17] (be aware that this definition is “larger” than the usual definition of a Fuchsian group, as a discrete subgroup of $PSL(2, \mathbb{R})$). If $\rho : \pi_1(M) \longrightarrow \pi_1(M)/N$ is the canonical surjection, and if we denote $\underline{u} = \rho(u)$, then $\pi_1(M)/N$ admits one of the following presentations, according to whether the base of M can be oriented or not:

$$\left\langle \underline{a}_1, \underline{b}_1, \dots, \underline{a}_g, \underline{b}_g, \underline{c}_1, \dots, \underline{c}_q, \underline{d}_1, \dots, \underline{d}_p \mid \underline{c}_j^{\alpha_j} = 1, \left(\prod_{i=1}^g [\underline{a}_i, \underline{b}_i] \right) \underline{c}_1 \cdots \underline{c}_q \underline{d}_1 \cdots \underline{d}_p = 1 \right\rangle,$$

$$\left\langle \underline{a}_1, \dots, \underline{a}_g, \underline{c}_1, \dots, \underline{c}_q, \underline{d}_1, \dots, \underline{d}_p \mid \underline{c}_j^{\alpha_j} = 1, \underline{a}_1^2 \underline{a}_2^2 \cdots \underline{a}_g^2 \underline{c}_1 \cdots \underline{c}_q \underline{d}_1 \cdots \underline{d}_p = 1 \right\rangle.$$

Such groups can be seen as the π_1 of compact Fuchsian 2-complexes (cf. [17]), or in a more modern terminology, as π_1^{orb} of compact 2-orbifolds whose singular sets consist only of a finite number of cone points. In this terminology, M inherits a structure of S^1 -bundle over such an orbifold (cf. [31]).

When M has a non-empty boundary, the quotient group $\pi_1(M)/N$ is particularly simple. Indeed, the last relation of the above presentations can be transformed into a relation of the form $\underline{d}_i = \omega$ (for some $i = 1, 2, \dots, p$), where ω is a word which does not involve the letter \underline{d}_i or its inverse; this allows the use of Tietze transformations, to discard this relation together with the letter \underline{d}_i . Hence, $\pi_1(M)/N$ is the free product of the cyclic groups generated by the remaining generators. The element \underline{d}_i is represented by the word ω , which is cyclically reduced and has length greater than 1 (in the sense of the free product decomposition).

5.2. Solving the boundary parallelism and 2-cosets problems

We solve the boundary parallelism and 2-cosets problems. In both cases the idea is to reduce them to the Fuchsian group $\pi_1(M)/N$, which easily provides solutions.

Proposition 5.1. *The boundary parallelism problem is solvable in the group of a Seifert fibered space with a non-empty boundary.*

Proof. We construct an algorithm which solves this problem. Note first that in the cases of $S^1 \times D^2$, $S^1 \times S^1 \times I$, and of the twisted I -bundle over $\mathbb{K}\mathbb{B}_2$, the solutions are obvious, so that we can exclude these cases.

Suppose T is a boundary subgroup of $\pi_1(M)$, generated by d_1, h , and that $u \in \pi_1(M)$ is the conjugate of an element of T , say $u \sim d_1^\alpha h^\beta$ for some integers α, β . Hence, $\underline{u} \sim \underline{d}_1^\alpha$ in $\pi_1(M)/N$. Since, $M \not\cong S^1 \times D^2$, M is ∂ -irreducible, and thus \underline{d}_1 has infinite order.

Since M has a non-empty boundary, $\pi_1(M)/N$ is a free product of cyclic groups. The element \underline{d}_1 is either a canonical generator, or a cyclically reduced word of length greater than 1. Now, using the conjugacy theorem in a free product (cf. [20]), one can easily determined whether $\underline{u} \sim \underline{d}_1^\alpha$ in $\pi_1(M)/N$, for some integer α , and eventually find $\underline{a} \in \pi_1(M)/N$ which conjugates \underline{d}_1^α into \underline{u} . If \underline{u} is not conjugate to \underline{d}_1^α for some α , then definitely u is not conjugate in $\pi_1(M)$ to an element of T . Else, if

$$\underline{u} = \underline{a}.\underline{d}_1^\alpha.\underline{a}^{-1} \quad \text{in } \pi_1(M)/N$$

then once $a \in \rho^{-1}(\underline{a})$ has been chosen, since $\ker \rho = N$,

$$u = a.d_1^\alpha.a^{-1}.h^\beta = a.d_1^\alpha h^{\varepsilon.\beta}.a^{-1} \quad \text{in } \pi_1(M)$$

for some $\beta \in \mathbb{Z}$ and $\varepsilon = \pm 1$ according to whether $a \in C$ or not. Using the word problem solution in $\pi_1(M)$, one can find $\beta \in \mathbb{Z}$, and thus the element $d_1^\alpha h^\beta \in T$ is conjugate with u in $\pi_1(M)$. According to Lemma 4.2, if $\alpha \neq 0$ or if the base of M is oriented, then it is the unique element of T conjugate to u . Otherwise, u is conjugate only to the two elements h^β and $h^{-\beta}$ of T . \square

Proposition 5.2. *The 2-cosets problem is solvable in the group of a Seifert fibered space with a non-empty boundary.*

Proof. In the cases of $S^1 \times S^1 \times I$ and $S^1 \times D^2$ the solutions are obvious, and in the case of the twisted I -bundle over $\mathbb{K}\mathbb{B}_2$, Lemma 4.3 implicitly provides a solution, so that we can exclude these cases.

Let T be a boundary subgroup of $\pi_1(M)$; it is a free abelian group of rank 2 generated by d_1, h . Let $u, v \in \pi_1(M)$; we begin by determining $C_{T,T}(u, v) = \{(t, t') \in T \times T \mid u = t.v.t'\}$. We first use Proposition 4.2 to decide whether $v \in T$ or not.

In the former case, since T is abelian, $C_{T,T}(u, v) = \{(uv^{-1}.t, t^{-1}) \mid t \in T\}$, so that we can now suppose that $v \notin T$. Suppose that $u = t.v.t'$, where $t = d_1^\alpha h^\beta$ and $t' = d_1^\gamma h^\delta$, for some integers $\alpha, \beta, \gamma, \delta$. Then,

$$\underline{u} = \underline{d}_1^\alpha . \underline{v} . \underline{d}_1^\gamma \quad \text{in } \pi_1(M)/N.$$

Since M is ∂ -irreducible, \underline{d}_1 has infinite order in $\pi_1(M)/N$. Moreover, no power of \underline{v} lies in $\langle \underline{d}_1 \rangle$, since, indeed $\langle \underline{d}_1 \rangle$ has a trivial root structure in $\pi_1(M)/N$, and we have supposed that $v \notin T$. Hence, since $\pi_1(M)/N$ is a free product of cyclic groups, one can use the normal form theorem (cf. [20]), to find, if any, such a couple (α, γ) . Thus, since $\ker \rho = N$,

$$u = d_1^\alpha . v . d_1^\gamma h^\theta \quad \text{in } \pi_1(M)$$

for some $\theta \in \mathbb{Z}$, that one can easily find using the solution to the word problem in $\pi_1(M)$. Hence we have found an element of $C_{T,T}(u, v)$, and using the Lemma 4.3, we determine $C_{T,T}(u, v)$ precisely.

Now suppose that we wish to determine $C_{T,T'}(u, v)$ for some distinct boundary subgroups T, T' . Suppose T, T' are, respectively, generated by d_1, h , and d_2, h . The elements \underline{d}_1 and \underline{d}_2 have infinite order in $\pi_1(M)/N$ and then, using the free product structure, we find, if any, a couple of integers (α, γ) , such that $\underline{u} = \underline{d}_1^\alpha . \underline{v} . \underline{d}_2^\gamma$. Then, we can apply the same process as before to find $C_{T,T'}(u, v)$ precisely. \square

5.3. Solving the conjugacy problem

In almost all cases, if M is a Seifert fiber space, $\pi_1(M)$ is biautomatic, and hence admits a solution to the conjugacy problem (cf. [24,25]); the remaining cases are those of (closed) Seifert fiber spaces modelled on NIL geometry, that is, S^1 -bundles over a flat orbifold with a non-zero Euler number. Anyway, the conjugacy problem in groups of Seifert fiber spaces can be easily solved by direct methods; this is neither difficult nor surprising, and we will only sketch a proof. The interested reader might refer to [29] for a detailed solution.

Conjugacy problem in $\pi_1(M)$ easily reduces to a conjugacy problem in $\pi_1(M)/N$ and to the problem consisting in determining canonical generators of the centralizer of any element of $\pi_1(M)/N$ (it can be cyclic, $\mathbb{Z} \oplus \mathbb{Z}$ or the group of the Klein bottle). For, suppose $u, v \in \pi_1(M)$ are given, and that we wish to decide whether $u \sim v$ or not. We use a solution to the conjugacy problem in $\pi_1(M)/N$ to decide whether $\underline{u} \sim \underline{v}$. If \underline{u} and \underline{v} are not conjugate in $\pi_1(M)/N$, then u, v are not conjugate in $\pi_1(M)$; else, there exists $\underline{a} \in \pi_1(M)/N$ such that $\underline{u} = \underline{a} \underline{v} \underline{a}^{-1}$, and if we choose $a \in \rho^{-1}(\underline{a})$, then $u = a v a^{-1} h^p$ in $\pi_1(M)$, for some $p \in \mathbb{Z}$ that one can determine using the solution to the word problem. Of course, if $p = 0$, u, v are conjugate in $\pi_1(M)$, but if $p \neq 0$ one cannot conclude anything at this point. To do so, one needs to determine the canonical generators of the centralizer $Z(\underline{v})$ of \underline{v} in $\pi_1(M)/N$. Suppose $Z(\underline{v})$ has generators x, y ; then, $v x v^{-1} x^{-1} = h^{n_1}$ and $v y v^{-1} y^{-1} = h^{n_2}$ in $\pi_1(M)$ for integers n_1, n_2 that one can determine. Then one can easily see that u and v will be conjugate in $\pi_1(M)$ exactly when p, n_1, n_2 satisfy arithmetic relations, which depend only on the isomorphism class of $Z(\underline{v})$, as well as on the memberships of v, x, y of the canonical subgroup of $\pi_1(M)$.

The problem of determining the centralizer of an element of $\pi_1(M)/N$, and the conjugacy problem in $\pi_1(M)/N$ can be easily solved, because $\pi_1(M)/N$ is either finite ($M \approx S^1 \times S^2$, or $\mathbb{P}^3 \# \mathbb{P}^3$ or modelled on

\mathbb{S}^3), word-hyperbolic $(\mathbb{H}^2 \times \mathbb{E}^1, \widetilde{SL}(2, \mathbb{R}))$, or a Bieberbach group (NIL, \mathbb{E}^3) . Thus the conjugacy problem can be solved in $\pi_1(M)$.

6. The case of a hyperbolic piece

In this section we give solutions to the needed decision problems in the group of a hyperbolic piece. A solution to the conjugacy problem is already well-known, according to the following result, which is a direct implication of Theorem 11.4.1 (geometrically finite implies biautomatic) of [8].

Theorem 6.1. *The group of a hyperbolic 3-manifold with finite volume is biautomatic and hence has a solvable conjugacy problem.*

The two remaining decision problems, namely the boundary parallelism problem and the 2-coset problem, will be solved using different approaches. The solution to the boundary parallelism problem will involve on the one hand word-hyperbolic group theory and on the other Thurston's surgery theorem in the spirit of Sela [34], while the 2-coset problem will involve a relatively hyperbolic group theory in the sense of Farb [9].

We first conduct some reviews (far from complete) on word hyperbolic groups, in order to recall elementary concepts and to fix notations.

6.1. Reviews on hyperbolic groups

To a group G with a fixed finite generating set X , one associates the *Cayley graph* $\Gamma = \Gamma(G, X)$, which is a locally finite directed labelled graph, by choosing a vertex \bar{g} for each element $g \in G$, and for all $g \in G$ and $s \in X \cup X^{-1}$ an edge with label s , going from \bar{g} to $\overline{g.s}$. To make Γ a metric space we assign to each edge the length 1, and we define the distance between two points to be the length of the shortest path joining them. Together with this metric, Γ becomes a proper geodesic space. Since vertices of Γ are in 1–1 correspondence with elements of G , the group G inherits a metric d_G , called *the word metric*. For an element $\omega \in G$, we denote $|\omega| = d_G(1, \omega) = d_\Gamma(\bar{1}, \bar{\omega})$, while the length of a word ω on the canonical generators will be noted by $\text{lg}(\omega)$. Remark that the group G acts on the left naturally by isometries on its Cayley graph Γ .

A finite path γ in Γ comes equipped with a label which is a word on the alphabet $X \cup X^{-1}$, naturally obtained by concatenating the labels of its edges. Given a vertex v_0 in Γ , finite paths of Γ starting from v_0 are in one-to-one correspondence with words on the generators. We will often make no distinction between a finite path and its label, as well as between an element of G and a vertex of Γ .

A geodesic metric space is said to be δ -hyperbolic if there exists $\delta \geq 0$ such that for any geodesic triangle (xyz) , each of its geodesics, for example $[x, y]$, stays in a δ -neighborhood of the union of the two others, $[y, z] \cup [x, z]$. Given a finite generating set X of G , the group G is said to be δ -hyperbolic (resp. hyperbolic) if its Cayley graph $\Gamma(G, X)$ is δ -hyperbolic (resp. δ -hyperbolic for some $\delta \geq 0$). It turns out that the property of being hyperbolic does not depend on the choice of a finite generating set X of G . If $\Gamma(G, X)$ is δ -hyperbolic, then $\Gamma(G, Y)$ is δ' -hyperbolic; moreover, once we know a set of words on X representing the elements of Y , we can easily give a bound on δ' , in terms of δ and of the maximal length of these words (cf. [3]).

Hyperbolic groups, introduced by Gromov [13] to generalize fundamental groups of closed negatively curved riemanian manifolds, have since been largely studied and implemented. It turns out that they admit very elegant algebraic properties, as well as a particular efficiency in algorithmic processes. For example they have solvable word and conjugacy problems, which can be solved, respectively, in linear and sub-quadratic times. For basic facts about hyperbolic groups, and usual background, we refer the reader to the reference books [12,13,11,3].

6.2. Solution to the boundary parallelism problem

To give a solution to the boundary parallelism problem, we will make use of the word-hyperbolic group theory, and of Thurston's hyperbolic surgery theorem ([37], see also [1, Theorem E.5.1]).

Theorem 6.2 (*Thurston's hyperbolic surgery theorem*). *Let M be a hyperbolic finite volume 3-manifold with a non-empty boundary. Then almost all manifolds obtained by Dehn filling on M are hyperbolic.*

Remarks. (1) The sense of “almost all” should be interpreted with caution. It means that if all the surgery coefficients are large enough, then the manifold obtained is hyperbolic. If M has only one boundary component, then “almost all” means “all but a finite number”.

(2) Let M be a hyperbolic 3-manifold with a non-empty boundary, and let N be a closed hyperbolic 3-manifold obtained by Dehn filling on M . Its fundamental group $\pi_1(N)$ is a cocompact (torsion free) discrete subgroup of $PSL(2, \mathbb{C})$, and thus is hyperbolic in the sense of Gromov. Note $\rho : \pi_1(M) \twoheadrightarrow \pi_1(N)$, the canonical epimorphism. Suppose T is a boundary subgroup of $\pi_1(M)$, that is, T is any maximal parabolic subgroup. Then, since the hyperbolic structure on N extends hyperbolic structures on the solid tori used in the surgery, the cores of surgery are geodesics of N and necessarily $\rho(T)$ must be cyclic infinite.

The underlying idea (belonging to Sela) is to obtain two closed hyperbolic 3-manifolds N_1 and N_2 by Dehn filling on M , and to use algorithms in $\pi_1(N_1)$ and $\pi_1(N_2)$ to provide a solution to the boundary parallelism problem in $\pi_1(M)$. Suppose $\rho_1 : \pi_1(M) \twoheadrightarrow \pi_1(N_1)$ and $\rho_2 : \pi_1(M) \twoheadrightarrow \pi_1(N_2)$ are the canonical epimorphisms, and suppose one wishes to decide for some $\omega \in \pi_1(M)$ and some boundary subgroup $T \subset \pi_1(M)$ whether ω is conjugate to an element of T or not. Deciding whether $\rho_i(\omega)$ is conjugate in $\pi_1(N_i)$ to an element of $\rho_i(T)$ (for $i = 1, 2$) will provide a solution in $\pi_1(M)$. We first need to establish the following lemma.

Proposition 6.1. *Let G be a torsion-free δ -hyperbolic group, and H a cyclic subgroup of G . Then, an arbitrary element of G can be conjugate to at most one element of H . Moreover, there exists an algorithm, which decides for any non-trivial element $\omega \in G$, if ω is conjugate in G to an element of H , and finds an eventual conjugate of ω in H .*

Proof. If $H = \{1\}$ the conclusion comes obviously with a solution to the word problem, so that we will further suppose that H is non-trivial.

We first prove the former part of the assumption. Since h has infinite order, it fixes two distinct points h^- and h^+ in the boundary $\partial\Gamma$ of the Cayley graph. Suppose that ω is conjugate with two distinct elements of $H = \langle h \rangle$, say h^p and h^q ; then, there exists $\alpha \in G$ such that $\alpha.h^p.\alpha^{-1} = h^q$. Necessarily the action of α on $\partial\Gamma$ must preserve h^- , h^+ . Hence [3, Proposition 7.1] α lies in a finite extension of H . In particular,

$\exists r > 0, s > 0$, such that $\alpha^r = h^s$. Then on the one hand $\alpha.h^{p.s}.\alpha^{-1} = (\alpha.h^p.\alpha^{-1})^s = h^{q.s}$, and on the other $\alpha.h^{p.s}.\alpha^{-1} = \alpha.\alpha^{p.r}.\alpha^{-1} = \alpha^{p.r} = h^{p.s}$, which implies $p = q$.

In order to prove the latter part of the assumption, we make use of the stable norm $\|g\|$ of an element $g \in G$ (cf. [3,13,Section 10.6]), defined as

$$\|g\| = \lim_{n \rightarrow \infty} \frac{|g^n|}{n}.$$

The limit exists since $0 \leq |g^{n+p}| \leq |g^n| + |g^p|$, and $\|g\|$ is indeed the infimum of $\{|g^n|/n; n > 0\}$. It can be seen easily that the stable norm is invariant by conjugation, that is, if $u = a.v.a^{-1}$, $\|u\| = \|v\|$ (note that $||u^n| - |v^n|| \leq 2.|a|$, divide by n , and make n go to infinity).

Now suppose that $\omega \sim h^t$ for some $t > 0$. Considering a subsequence with indices $t.n$, one has

$$\|h\| = \lim_{n \rightarrow \infty} \frac{|h^{tn}|}{tn} = \frac{1}{t} \lim_{n \rightarrow \infty} \frac{|h^{tn}|}{n}.$$

But since $\omega \sim h^t$,

$$\lim_{n \rightarrow \infty} \frac{|h^{tn}|}{n} = \|h^t\| = \|\omega\|$$

and hence

$$\|h\| = \frac{\|\omega\|}{t}.$$

Now the key point is that there exists a computable constant $K > 0$, which only depends on δ , such that any element g of infinite order satisfies $\|g\| \geq K$ (cf. [13,7, 13, Remark p. 254; and 7, Proposition 3.1], for a sketch of a proof). So we finally obtain

$$\frac{|\omega|}{t} \geq \frac{\|\omega\|}{t} \geq \|h\| \geq K > 0$$

which shows that

$$t \leq \frac{|\omega|}{K}.$$

It is now sufficient to use a solution to the word problem to compute $|\omega|$, and to decide with a solution to the conjugacy problem whether ω is conjugate with h^t for some $t \in \mathbb{Z}$, with modulus $|t| < |\omega|/K$. \square

We can now give the solution to the boundary parallelism problem in the group of a finite volume hyperbolic 3-manifold with a non-empty boundary.

Theorem 6.3. *The boundary conjugacy problem is solvable in the group of a finite volume hyperbolic 3-manifold with a non-empty boundary.*

Proof. Let M be a finite volume hyperbolic 3-manifold with a non-empty boundary, and $\mathcal{T} \subset \partial M$ be a (toroidal) boundary component. Enumerate all closed 3-manifolds obtained by Dehn filling on M : each one corresponds (once bases are given) to a couple of coprime integers for each of the components of ∂M . While continuing the enumeration, conduct in parallel a computation of a finite presentation of $\pi_1(N)$ for

each closed manifold N obtained, and then apply the pseudo-algorithm appearing in [28]. It checks the hyperbolicity of $\pi_1(N)$, and if so stops yielding a constant δ such that $\pi_1(N)$ is δ -hyperbolic. Pursue these parallel processes until you have found two groups $\pi_1(N_1)$, $\pi_1(N_2)$ —obtained by distinct surgery slopes on \mathcal{T} —which are hyperbolic, and if so, stop all process. According to Thurston’s hyperbolic surgery theorem, the general process will terminate. Denote T the boundary subgroup of $\pi_1(M)$ associated with \mathcal{T} and $g_1, g_2 \in T$ the respective elements associated with surgery slopes (up to inverses) on \mathcal{T} of $\pi_1(N_1)$, $\pi_1(N_2)$.

Once $\pi_1(N_1)$, $\pi_1(N_2)$ and constants of hyperbolicity are given, we can apply a process which allows to decide for any arbitrary element ω , if ω is conjugate to an element of T , and find all such conjugate elements. This process is described below.

The element $g_1 \in \pi_1(M)$ is the class of a simple closed curve on \mathcal{T} , and hence can be completed to form a base g_1, h_1 of $T = \mathbb{Z} \oplus \mathbb{Z}$. Now consider the canonical epimorphism $\rho_1 : \pi_1(M) \rightarrow \pi_1(N_1)$, $\rho_1(T)$ is a cyclic infinite subgroup of $\pi_1(N_1)$ generated by $\rho_1(h_1) = h$. Since N_1 is hyperbolic, $\pi_1(N_1)$ has no torsion. Let ω be an arbitrary element of $\pi_1(M)$. We wish to decide whether ω is conjugate to an element of T . If $\rho_1(\omega)$ is non-trivial, then we can use Proposition 6.1 to find at most one element h^p conjugate to $\rho_1(\omega)$ in $\pi_1(N_1)$. If $\rho_1(\omega) = 1$, it is obviously conjugate to an element of $\rho_1(T)$. Hence possible conjugates of ω in T must be of the form $h_1^p \cdot g_1^n$, for some $n \in \mathbb{Z}$, where p is given. Look at the Cayley graph of T as naturally embedded in the universal cover \mathbb{R}^2 of the torus \mathcal{T} . Then eventual conjugates in T of ω must lie on the line, with slope g_1 crossing $\overline{h_1^p}$. But applying the same process in $\pi_1(N_2)$, conjugates of ω in T must lie at the same time on the line with slope g_2 crossing some given point. Hence, since the two slopes have been chosen to be distinct they must be non-collinear, and one easily finds (by resolving a system of two linear equations) at most one element of T which can be conjugate with ω in $\pi_1(M)$. Applying the solution to the conjugacy problem in $\pi_1(M)$, one determines which element of T , if any, is conjugate with ω . \square

This method can also be applied to solve the 2-coset problem, but to do so one needs a refinement of Thurston’s surgery theorem, which asserts that there exists a sequence of closed hyperbolic manifolds converging to M for the geometric topology. But this case is difficult, and gives a solution that is much less satisfactory, since the surged closed hyperbolic manifolds do depend on the element $\omega \in G$ that one considers (cf. [29, Section 4.3]). A better approach uses a relatively hyperbolic groups theory in the sense of Farb, as seen in the following two sections. We first recall in the next section elementary facts on relatively hyperbolic groups.

6.3. Reviews on relatively hyperbolic groups

This section follows Section 6.1. Consider a finitely generated group G , and finitely generated subgroups H_1, H_2, \dots, H_n of G . Start from the Cayley graph Γ of G , and for each left coset $g.H_i$ add a new vertex $v(g.H_i)$, as well as an edge $e(g.h)$ with length $1/2$ from each vertex $\overline{g.h}$ such that $h \in H_i$, to $v(g.H_i)$. These new vertices and edges will be called *special vertices* and *special edges*. This gives rise to a new graph $\hat{\Gamma}$, called the *coned-off Cayley graph* (which does not have to be locally finite), together with a natural metric which makes $\hat{\Gamma}$ a (non-necessarily proper) geodesic metric space. Note that Γ naturally embeds in $\hat{\Gamma}$, but that this embedding does not (except in the trivial case involving trivial subgroups) preserve lengths.

The group G is said to be δ -hyperbolic relative to H_1, H_2, \dots, H_n , if its coned-off Cayley graph $\hat{\Gamma}$ is a δ -hyperbolic geodesic space. It turns out that this definition does not depend on the choice of a finite generating set of G .

Suppose X is a finite generating set of G , and that one knows for each H_i a finite set of words $S_i = \{y_{i,j} \mid j\}$ on X generating H_i . Given a path w in Γ , there is a usual way of finding a corresponding path \hat{w} in $\hat{\Gamma}$. Processing from left to right, one searches in w a maximal sub-word on the family S_i . For each maximal sub-word, say z_i on S_i , z_i goes from the vertex \bar{g} to $\bar{g} \cdot z_i$; replace this path with one edge from \bar{g} to the special vertex $v(gH_i)$, followed by an edge from $v(gH_i)$ to $\bar{g} \cdot z_i$ (we make no distinction between a path and its label in Γ). Proceed in this manner until it is impossible; obviously the process will halt. This replacement gives a surjective map $\Gamma \twoheadrightarrow \hat{\Gamma}$ which from a path ω in Γ gives a path that we shall denote $\hat{\omega}$ in $\hat{\Gamma}$. If $\hat{\omega}$ passes through some special vertex $v(gH_i)$, we say that ω (or $\hat{\omega}$) *penetrates* the coset gH_i , or equivalently that ω (or $\hat{\omega}$) penetrates the special vertex $v(gH_i)$.

The path w of Γ is said to be a *relative geodesic*, if \hat{w} is a geodesic of $\hat{\Gamma}$. The path w is said to be a *relative quasi-geodesic*, if \hat{w} is a quasi-geodesic. A path w in Γ (or \hat{w} in $\hat{\Gamma}$) is said to be *without backtracking*, if for every coset gH_i that w penetrates, \hat{w} does not return to gH_i after leaving gH_i . Obviously a relative geodesic is without backtracking.

To proceed efficiently with relative hyperbolic groups, one needs a more restricted property, the bounded coset penetration property:

Bounded coset penetration property (or BCP property for short): Let G be a group hyperbolic relative to H_1, H_2, \dots, H_n . Given finite generating sets for G, H_1, \dots, H_n , G is said to satisfy the bounded coset penetration property, if for every $P \geq 1$, there is a constant $c = c(P) > 0$, so that if u and v are relative P -quasigeodesics in Γ without backtracking, and with $d_\Gamma(\bar{u}, \bar{v}) \leq 1$, then the following conditions hold:

- if u penetrates a coset gH_i but v does not penetrate gH_i , then u travels a Γ -distance of at most c in gH_i .
- If u and v both penetrate a coset gH_i , then the vertices at which u and v first enter gH_i lie at a Γ -distance of at most c from each other, and similarly for the vertices at which u and v last exit gH_i .

It turns out that verifying the BCP property does not depend on the choice of finite generating sets of G, H_1, H_2, \dots, H_n (cf. [9]).

Our motivation for introducing these notions comes from the following result [9, Theorem 5.1].

Theorem 6.4. *The fundamental group of a complete finite volume negatively curved riemannian manifold is hyperbolic relative to the set of its cusp-subgroups and satisfies the BCP property. In particular, the same conclusions hold for fundamental groups of finite volume hyperbolic 3-manifolds relative to their boundary subgroups.*

6.4. Solution to the 2-cosets problem

We now give a solution to the 2-coset problem in the group of a hyperbolic 3-manifold with finite volume. We make use of the fact that $\pi_1(M)$ is hyperbolic relative to its boundary subgroups, and satisfies the BCP property (in fact only the last property is necessary). The key point is the following result:

Fig. 5.

Lemma 6.1. *Let G be a hyperbolic group relative to its subgroups H_1, H_2, \dots, H_n , which satisfies the BCP property. Let $u, v \in G$, $i, j \in \{1, 2, \dots, n\}$ such that if $i = j$, then u or v does not lie in H_i , and suppose that there exist $c_1 \in H_i, c_2 \in H_j$ such that $u = c_1.v.c_2$ in G .*

Then, there exists a constant K which only depends on $\lg(u), \lg(v)$ and on constants related to the relative hyperbolic structure, such that c_1 and c_2 have length at most K for the word metric d_G .

Proof. We suppose finite generating sets are given, and will observe the Cayley graph of Γ as (non-isometrically) embedded in the coned-off Cayley graph $\hat{\Gamma}$. Consider words u and v such that $u = c_1.v.c_2^{-1}$ in G , for some $c_1 \in H_1$ and $c_2 \in H_2$ (eventually $H_1 = H_2$). We choose the words representing c_1, c_2 such that they are labels of relative geodesics; hence, \hat{c}_1 is a path of $\hat{\Gamma}$ of length 1 starting from $\bar{1}$ going through a special edge to the special vertex $v(H_1)$ and going back through a special edge to the vertex \bar{c}_1 , and similarly \hat{c}_2 is a path of length 1 starting from \bar{u} and ending in $\overline{u.c_2}$, which crosses the special vertex $v(u.H_2)$. The relation $u = c_1.v.c_2^{-1}$ in G gives rise to a quadrilateral in Γ , with vertices $\bar{1}, \bar{c}_1, \bar{u}, \overline{u.c_2}$ and edges labelled with the words c_1, v, c_2, u , such that those with labels c_1, c_2 are relative geodesics.

For any path $\alpha \subset \hat{\Gamma}$ and any positive integer $t \leq \lg(\alpha)$, we will denote $\alpha(t)$ the vertex of α such that the sub-path of α from its origin to $\alpha(t)$ exactly has length t . In the following u and $c_1.v$ stand for the paths in $\hat{\Gamma}$ with labels, respectively, u and v , going, respectively, from $\bar{1}$ to \bar{u} and from \bar{c}_1 to $\bar{c}_1.v$. Denote ω_1 a relative geodesic path starting from the origin $u(0) = c_1(0) = \bar{1}$ of u and ending in $c_1.v(1)$, ω_2 a relative geodesic path starting from $u(1)$ and ending in $c_1.v(1)$, ω_3 a relative geodesic path starting from $u(1)$ and ending in $c_1.v(2)$, etc. . . until u or $c_1.v$ does not have any vertex left, for suppose $p = \lg(u) \leq \lg(v)$, then ω_{2p-1} goes from $u(p-1)$ to $c_1.v(p)$, and then we denote ω_{2p} the relative geodesic from $u(p-1)$ to $c_1.v(p+1)$, $\omega(2p+1)$ the relative geodesic going from $u(p-1)$ to $c_1.v(p+2)$, etc. . . , until the last vertex $\overline{u.c_2}$ of $c_1.v$. Hence, finally we obtain $k = \lg(u) + \lg(v) + 1$ relative geodesics $c_1, \omega_1, \omega_2, \dots, \omega_{k-2}$ and c_2 , such that two successive ones have the same origins and their extremities lie at a distance 1, one to the other, or converse (Fig. 5).

Now suppose that c_1 has length L ; then, c_1 travels a distance L in the special vertex $v(H_1)$. According to the BCP property, if c_1 has a length greater than C , then ω_1 must also cross the special vertex $v(H_1)$, if c_1 has a length greater than $3C$, ω_1 travels $v(H_1)$ a distance greater than C , and ω_2 also crosses $v(H_1)$, etc. . . , and if c_1 has a length greater than $C.(2(\lg(u) + \lg(v)) + 1)$, c_2 must also cross $v(H_1)$. But since by construction c_2 only crosses $v(u.H_2)$ this would imply $H_1 = u.H_2$, and hence $u \in H_1$ and $H_1 = H_2$ which

contradicts the hypothesis. Thus we have shown that $|c_1| \leq C \cdot (2(\lg(u) + \lg(v)) + 1)$; the same argument applies to show that $|c_2| \leq C \cdot (2(\lg(u) + \lg(v)) + 1)$. \square

Proposition 6.2. *The group of a hyperbolic 3-manifold with a non-empty boundary and finite volume has a solvable 2-cosets problem.*

Proof. Using the last lemma, it suffices to consider all the couples of elements lying in the closed ball of G with origin 1 and radius $C \cdot (2(\lg(u) + \lg(v)) + 1)$, and to use the solution to the word problem to find a possible couple (c_1, c_2) with $u = c_1.v.c_2$. For any such couple, use the Proposition 4.2 to decide whether $c_1 \in H_1$ and $c_2 \in H_2$, and conclude. \square

7. Solutions to the conjugacy problem in the remaining cases

The remaining cases are:

- An $S^1 \times S^1$ -bundle over S^1 . Its group is an HNN extension of $\mathbb{Z} \oplus \mathbb{Z}$ with associated isomorphism $\phi : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ lying in $SL(2, \mathbb{Z})$; it is indeed the semi-direct product $\mathbb{Z} \rtimes_{\phi} \mathbb{Z}$. Moreover, we can suppose that ϕ is Anosov (two different irrational eigenvalues), for if ϕ is periodic (two complex conjugate eigenvalues, p -roots of the unity) or reducible (one eigenvalue 1 or -1), one observes easily that the manifold admits a Seifert fibration.
- A manifold obtained by gluing two twisted I -bundles over $\mathbb{K}\mathbb{B}_2$ along their (toroidal) boundary. Its group is an amalgamated product of two copies of $\langle a, b \mid aba^{-1} = b^{-1} \rangle$ along the two copies of the subgroup $\langle a^2, b \rangle$, with associated isomorphism lying in $SL(2, \mathbb{Z})$.

In each case, using the conjugacy theorem in amalgams (cf. [20,29]) or HNN extensions (cf. [19,29]), the conjugacy problem reduces to matrix equations in $SL(2, \mathbb{Z})$, which can be easily solved.

Suppose M is an $S^1 \times S^1$ -bundle over S^1 , with an Anosov associated gluing-map. Then $\pi_1(M) = \mathbb{Z} \rtimes_{\phi} \mathbb{Z}$ where $\phi \in SL(2, \mathbb{Z})$ has two distinct irrational eigenvalues. If we denote G the first factor of the semi-direct product, and t a generator of the second factor, each element of $\pi_1(M)$ can be uniquely written in the canonical form $u.t^p$, where $u \in G$ and $p \in \mathbb{Z}$. Now consider two elements $u.t^p$ and $v.t^q$ in $\pi_1(M)$, and suppose that they are conjugate. Then considering the homomorphism from $\pi_1(M)$ to \mathbb{Z} which sends $\mathbb{Z} \oplus \mathbb{Z}$ to 0 and t to a generator of \mathbb{Z} , necessarily $p = q$.

Suppose first that $p = q = 0$. Then u and v are conjugate in $\pi_1(M)$ if and only if there exists $n \in \mathbb{Z}$ such that $u = \phi^n(v)$. To decide this, consider a base of $G = \mathbb{Z} \oplus \mathbb{Z}$ constituting eigenvectors of ϕ . Then the above equation is equivalent to the system: $u_1 = \lambda_1^n.v_1$ and $u_2 = \lambda_2^n.v_2$, where $u = (u_1, u_2)$, $v = (v_1, v_2)$ and λ_1, λ_2 are the eigenvalues of ϕ . The elements $u_1, u_2, v_1, v_2, \lambda_1, \lambda_2$ lie in the extension field $\mathbb{Q}(\sqrt{\Delta})$ where Δ stands for the discriminant of the characteristic polynomial of the matrix associated to ϕ according to the canonical basis of $\mathbb{Z} \oplus \mathbb{Z}$. This system can be easily solved providing a solution in this case.

Suppose now that $p = q$ are distinct from 0. Using the conjugacy theorem in an HNN extension, one sees easily that $u.t^p$ and $v.t^p$ are conjugate if and only if there exists $c \in G$ such that $u^{-1}v = c.\phi^p(c)^{-1}$ up to cyclically conjugating $v.t^p$. Let us first see how to decide whether there exists $c \in G$ such that $u^{-1}v = c.\phi^p(c)^{-1}$. Consider the canonical base of $\mathbb{Z} \oplus \mathbb{Z}$; in this base $u^{-1}v = (n_1, n_2)$, and ϕ^p has an

associated matrix:

$$\mathcal{M} = \text{Mat}(\phi^p) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. We look for $c = (x, y) \in \mathbb{Z} \oplus \mathbb{Z}$ such that $u^{-1}v = c \cdot \phi^p(c)^{-1}$. Then,

$$\begin{aligned} u^{-1}v = c \phi^p(c)^{-1} &\Leftrightarrow \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ &\Leftrightarrow \begin{cases} n_1 = (1 - \alpha)x - \beta y \\ n_2 = -\gamma x + (1 - \delta)y. \end{cases} \end{aligned}$$

This system has the determinant: $\det(\text{Id} - \mathcal{M})$, which is the value in one of the characteristic polynomial of \mathcal{M} . But since ϕ is Anosov, ϕ^p is also Anosov and hence does not admit 1 as an eigenvalue; thus, the system admits a unique solution $(x, y) \in \mathbb{Q} \times \mathbb{Q}$, and if x and y both lie in \mathbb{Z} , then $u \cdot t^p$ and $v \cdot t^p$ are conjugate in $\pi_1(M)$. To conclude apply the same process with all the $p + 1$ cyclic conjugates of $v \cdot t^p$ (they have the form $\phi^q(v) \cdot t^p$ for $q = 0, 1, \dots, p$). According to the conjugacy theorem if one does not find that $u \cdot t^p \sim v \cdot t^p$ in this way, then they are definitely not conjugate. \square

Suppose M is obtained by gluing two twisted I -bundles over S^1 . Denote $N_1 \approx N_2$ the two I -bundles, and ϕ the gluing homeomorphism $\phi : \partial N_1 \rightarrow \partial N_2$. We denote $\pi_1(N_1) = \langle a_1, b_1 | a_1 b_1 a_1^{-1} = b_1^{-1} \rangle$, $\pi_1(N_2) = \langle a_2, b_2 | a_2 b_2 a_2^{-1} = b_2^{-1} \rangle$, for $i = 1, 2$, $H_i = \langle a_i^2, b_i \rangle$, and $\varphi : H_1 \rightarrow H_2$ the isomorphism induced by ϕ . By fixing respective basis $(a_1)^2, b_1$ and $(a_2)^2, b_2$ of the free abelian groups of rank two H_1 and H_2 , φ can be seen as an element of $SL(2, \mathbb{Z})$. With these bases, denote also φ_1 and φ_2 the respective automorphisms of H_1 and H_2 associated to the matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The group $\pi_1(M)$ is the amalgamated product of $\pi_1(N_1)$ and $\pi_1(N_2)$ along φ . Each element of $\pi_1(M)$ can be cyclically reduced in an element which either lies in a factor $\pi_1(N_1)$ or $\pi_1(N_2)$, or is of the form: $U = (a_1 a_2)^n \cdot u$, with $n \in \mathbb{Z}$ and $u \in H_1 = H_2$. We consider $U, V \in \pi_1(M)$, and wish to decide whether they are conjugate. According to the conjugacy theorem in amalgams (cf. [20]), if $U \sim V$, then up to cyclic conjugations, either U, V both lie in a factor, or there exists $n \in \mathbb{Z}$, and $u, v \in H_1 = H_2$, such that $U = (a_1 a_2)^n u$ and $V = (a_1 a_2)^n v$.

Suppose first that U and V both lie in a factor, say $U = a_1^{n_1} b_1^{m_1}$ and $V = a_1^{n_2} b_1^{m_2}$ lie in $\pi_1(N_1)$; we need to decide whether they are conjugate in $\pi_1(N_1)$. It is an easy exercise to show that $U \sim V$ in $\pi_1(N_1)$ exactly if either $n_1 = n_2$ and $m_1 = \pm m_2$, or $n_1 = n_2$ is odd and $m_1 = m_2 \bmod 2$, which can be easily checked. If U and V are not conjugate in $\pi_1(N_1)$, or if they lie in distinct factors, then according to the conjugacy theorem, to be conjugate in $\pi_1(M)$ they must necessarily be conjugate in their respective factors $\pi_1(N_i)$ to elements of H_i ($i = 1$ or 2), and thus, since H_i is normal in $\pi_1(N_i)$, they must lie in H_i . By eventually considering $\varphi^{-1}(U)$ or $\varphi^{-1}(V)$ rather than U or V , we will suppose that both U and V lie in $H_1 \subset \pi_1(N_1)$. Applying the conjugacy theorem in this case one can see easily that $U \sim V$ in $\pi_1(M)$ exactly if there exists an integer n , such that (equation (*)): $(n_2, m_2) = (\varphi^{-1} \circ \varphi_2 \circ \varphi \circ \varphi_1)^n(n_1, m_1)$. Suppose

$$\text{Mat}(\varphi) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then an easy calculation shows that

$$\mathcal{M} = \text{Mat}(\varphi^{-1} \circ \varphi_2 \circ \varphi \circ \varphi_1) = \begin{pmatrix} \alpha\delta + \beta\gamma & -2\beta\delta \\ -2\alpha\gamma & \alpha\delta + \beta\gamma \end{pmatrix}$$

and that the associated endomorphism is either Anosov or reducible according to whether $\beta\gamma \neq 0, -1$ or not. When it is Anosov one can diagonalize the matrix \mathcal{M} , and then easily decide whether a solution n to (*) exists, that is, whether $U \sim V$ in $\pi_1(M)$ or not. When it is reducible, the matrix \mathcal{M} can only be triangulized, but in this form its diagonal consists only either of 1 or of -1 , and \mathcal{M}^n has a very simple form which can be easily computed and used to solve (*), concluding this case.

Suppose now that neither U nor V lies in a factor, and that they are conjugate in $\pi_1(M)$; then for some $p \in \mathbb{Z}$, $U = (a_1 a_2)^p \cdot u$ and $V = (a_1 a_2)^p \cdot v$, with $u, v \in H_1$. We note $\psi = \varphi^{-1} \circ \varphi_2 \circ \varphi \circ \varphi_1$; applying the conjugacy theorem one obtains that $U \sim V$ in $\pi_1(M)$ if and only if, up to cyclic conjugation of V , there exists $c \in H_1$ such that $vu^{-1} = \psi^p(c) \cdot c^{-1}$ (the $p + 1$ cyclic conjugates of V have the form $(a_1 a_2)^p \psi^k(v)$ for $k = 0, 1, \dots, p$). This condition is analogous to a condition treated above in the case of an $S^1 \times S^1$ -bundle over S^1 , and can be solved in the same way. There is nevertheless a difference: ψ^p can be Anosov but also reducible; as above, this last case can be easily implemented and does not present any difficulty. \square

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Note added in proof

In a recent paper (2005, to be submitted), the author has proven that non-oriented geometrizable 3-manifolds also have a group with solvable conjugacy problem: hence all geometrizable 3-manifold groups have a solvable conjugacy problem.

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